Attempt questions 1, 4 (a), (c), (e), (g), (h), 9 (b), (c), (j), 11 (d), (f) before your tutorial class in week 3.

A. Fractions

1. Frank needs to get to a meeting scheduled for 11 am. He leaves home at 8 am. He takes a quarter of an hour to walk to the bus stop, where he waits half an hour for the bus to arrive. After a trip of one and three quarter hours he reaches his destination.

(a) How long (in hours) was his total journey?
(b) Does he reach his destination in time for the meeting?
(c) Frank would like to have breakfast at a nearby cafe at his destination. Breakfast usually takes half an hour to order and eat. Will he have time for breakfast?

Solution.
(a) \( \frac{1}{4} + \frac{1}{2} + 1 \frac{3}{4} = 2 \frac{1}{2} \) hours
(b) He reaches his destination at 10:30 am, so he is in time for his meeting.
(c) Yes he does — he has half an hour before his meeting.

2. What is the sum of \( \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} \) and \( \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \)?

Solution. If we re-arrange the terms in the sum then the corresponding positive and negative terms cancel out.

\[
\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} + \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} = 1
\]

3. If \( 2 \frac{1}{4} \) cakes are served in slices of \( \frac{1}{8} \) of a cake, how many slices are served?

Solution.
\[
2 \frac{1}{4} \div \frac{1}{8} = 9 \times \frac{8}{1} = 18
\]

4. Express the following fractions as decimals.

(a) \( \frac{1}{2} = 0.5 \)  
(b) \( \frac{1}{4} = 0.25 \)  
(c) \( \frac{2}{5} = 0.4 \)  
(d) \( \frac{1}{8} = 0.125 \)  
(e) \( \frac{3}{8} = 0.375 \)  
(f) \( \frac{5}{8} = 0.625 \)  
(i) \( \frac{1}{25} = 0.04 \)  
(j) \( \frac{3}{25} = 0.12 \)  
(k) \( \frac{1}{100} = 0.01 \)  
(l) \( \frac{1}{1000} = 0.001 \)  
(m) \( \frac{1}{50} = 0.02 \)  
(n) \( \frac{2}{5} = 0.4 \)  
(o) \( \frac{125}{1000} = 0.125 \)

5. Lian earns $2,000 a fortnight and spends $1200 of that. What proportion of his salary does he save?

Solution. Lian saves \( \frac{2}{5} \) of his salary.

6. Nadia buys a packet of 60 samosas for a tea party. She takes out half for her family. She takes the remaining to the party. On the way she meets a friend who takes a sixth of the samosas. A little further on she sees a hungry cat and gives it a fifth of the samosas she has. The she suddenly feels hungry and eats a quarter of the remaining samosas. How many does she arrive with at the tea party?

Solution. Nadia starts with half the samosas for her party, so she takes \( \frac{1}{2} \times 60 = 30 \) samosas. Her friend takes \( \frac{1}{6} \) of these, so she is left with \( \frac{5}{6} \times 30 = 25 \) samosas. She gives a cat \( \frac{1}{5} \) of these, so she is left with \( \frac{4}{5} \times 25 = 20 \) samosas. She eats \( \frac{1}{4} \) of these, so she is left with \( \frac{3}{4} \times 20 = 15 \) samosas. She arrives at the party with 15 samosas.
7. (Shannon Guest) I have a naan recipe that calls for \( \frac{42}{3} \) cups of flour for 14 servings of naan. I want to make one serving only. How much flour should I use?

Solution.

\[
\frac{\frac{42}{3}}{14} = \frac{14}{3} \times \frac{1}{14} = \frac{1}{3},
\]

so I need \( \frac{1}{3} \) cup of flour.

8. (Shannon Guest) You currently have one third of a tank of fuel and you need half a tank to get home for the weekend. Your car holds 60 litres of petrol. The price of petrol today is 119.4 c and you have $14 only. Do you have enough money to get home for the weekend? Do you think you also have enough to buy a Snickers for $1.50?

Solution. My car has one third tank of petrol, that is, \( \frac{1}{3} \times 60 = 20 \) litres. I need half a tank of petrol to get home, that is, \( \frac{1}{2} \times 60 = 30 \) litres. So I need another \( 30 - 20 = 10 \) litres of petrol, which will cost \( 10 \times 119.4 = 1194 \) c. Since I have $14, I have enough money to get home for the weekend. I am left with $14 - $11.94 = $2.06, so I have enough for a Snickers as well.

B. Equations

9. Solve the following equations for the unknown.

(a) \( x + 3 = 5 \)

Solution.

\[
\iff x + 3 = 5 \\
x + 3 - 3 = 5 - 3 \\
x = 2
\]

(b) \( 4y - 5 = y + 10 \)

Solution.

\[
4y - 5 = y + 10 \\
\iff 4y - 5 + 5 = y + 10 + 5 \\
\iff 4y = y + 15 \\
\iff 4y - y = y + 15 - y \\
\iff 3y = 15 \\
\iff \frac{3y}{3} = \frac{15}{3} \\
\iff y = 5
\]

(c) \( \frac{2z}{z + 2} = 4 \)

Solution. First we remove the bottom line from the LHS by multiplying both sides by \( z + 2 \).

\[
\iff \frac{2z}{z + 2} = 4 \\
\iff (z + 2) \frac{2z}{z + 2} = 4(z + 2) \\
\iff 2z = 4z + 8 \\
\iff 2z - 4z = 4z + 8 - 2z \\
\iff -2z = 8 \\
\iff z = -4
\]

(d) \( 8x - 4 = 16 \)

Solution.

\[
8x - 4 = 16 \\
\iff 8x - 4 + 4 = 16 + 4 \\
\iff \frac{8x}{8} = \frac{16}{8} \\
\iff x = \frac{20}{8} = \frac{5}{2}
\]

(e) \( x - \frac{1}{2} = 2 \)

Solution.

\[
x - \frac{1}{2} = 2 \\
\iff x = \frac{3}{2} = \frac{5}{2}
\]

(f) \( 3 - 2x = 4 \)

Solution.

\[
3 - 2x = 4 \\
\iff -3 + 3 - 2x = 4 - 3 \\
\iff -2x = 1 \\
\iff x = -\frac{1}{2}
\]

(g) \( 7x + 7 = 2(x + 1) \)

Solution.

\[
7x + 7 = 2(x + 1) \\
\iff 7x + 7 = 2x + 2 \\
\iff 7x + 7 - 2x = 2x + 2 - 7 - 2x \\
\iff 5x = -5 \\
\iff x = -1
\]

(h) \( \frac{2y - 3}{4} = \frac{6y + 7}{3} \)

Solution. The LCM(3, 4) = 12, so we first multiply both sides by 12 to remove
the denominator.
\[\iff 12 \frac{2y - 3}{4} = 12 \frac{6y + 7}{3}\]
\[\iff 3 (2y - 3) = 4 (6y + 7)\]
\[\iff 6y - 9 = 24y + 28\]
\[\iff 6y - 9 + 9 - 24y = 24y + 28 + 9 - 24y\]
\[\iff -18y = 37\]
\[\iff y = -\frac{37}{18}\]

(i) \(t = 2 - 2[2t - 3(1 - t)]\)

**Solution.** First we simplify the bracket term.

\[\iff t = 2 - 2[2t - 3 + t]\]
\[\iff = 2 - 2[5t - 3]\]
\[\iff = 2 - 10t + 6\]
\[\iff = 8 - 10t\]
\[\iff t + 10t = 8 - 10t + 10t\]
\[\iff 11t = 8\]
\[\iff t = \frac{8}{11}\]

(j) \(\frac{3}{2}(4a - 3) = 2[a - (4a - 3)]\)

**Solution.** First we simplify the bracket term on the RHS, and then multiply both sides by 2 to remove the denominator on the LHS.

\[\iff \frac{3}{2}(4a - 3) = 2[a - (4a - 3)]\]
\[\iff = 2[a - 4a + 3]\]
\[\iff = 2[3 - 3a]\]
\[\iff 2 \cdot \frac{3}{2}(4a - 3) = 2 \cdot 2(3 - 3a)\]
\[\iff 3(4a - 3) = 12 - 12a\]
\[\iff 12a - 9 = 12 - 12a\]
\[\iff 12a - 9 + 9 + 12a = 12 - 12a + 9 + 12a\]
\[\iff 24a = 21\]
\[\iff a = \frac{21}{24} = \frac{7}{8}\]

(k) \(\frac{x + 3}{x} = \frac{2}{5}\)

**Solution.** This time the LCM(5, x) = 5x, so we multiply both sides by 5x to remove denominators from both sides.

\[\iff 5x \cdot (x + 3) = 5x \cdot \frac{2}{5}\]
\[\iff 5(x + 3) = 2x\]
\[\iff 5x + 15 = 2x\]
\[\iff 3x = -15\]
\[\iff x = -5.\]

(l) \(\frac{1}{p - 1} = \frac{2}{p - 2}\)

**Solution.** First note that \(p \neq 1, 2\), since then we are dividing by zero. Then we first multiply both sides by \((p - 1)(p - 2)\) to remove denominators.

\[\iff (p - 1)(p - 2) \cdot \frac{1}{(p - 1)} = (p - 1)(p - 2) \cdot \frac{2}{(p - 2)}\]
\[\iff p - 2 = 2(p - 1)\]
\[\iff p - 2 - p = 2p - 2 + 2 - p\]
\[\iff 0 = p\]
\[\text{or } p = 0\]

(m) \(\frac{x}{2} - \frac{x}{5} = 1\)

**Solution.** LCM(2, 5) = 10 so

\[\iff 10 \cdot \frac{x}{2} - 10 \cdot \frac{x}{5} = 10\]
\[\iff 5x - 2x = 10\]
\[\iff 3x = 10\]
\[\iff x = \frac{10}{3}\]

(n) \(\frac{y - 4}{2} + \frac{y}{3} = 4\)

**Solution.** LCM (2, 3) = 6, so

\[\iff 6 \left(\frac{y - 4}{2} + \frac{y}{3}\right) = 6 \times 4\]
\[\iff 3(y - 4) + 2y = 24\]
\[\iff 3y - 12 + 2y = 24\]
\[\iff 5y = 36\]
\[\iff y = \frac{36}{5}\]

(o) \(\frac{2}{y} + 4 = \frac{3}{y}\)

**Solution.**

\[\iff y \left(\frac{2}{y} + 4\right) = \frac{3}{y}\]
\[\iff 2 + 4y = 3\]
\[\iff 4y = 1\]
\[\iff y = \frac{1}{4}\]

(p) \(\frac{2 - y}{3} - \frac{1 - y}{2} = y\)

**Solution.**

\[\iff 6 \left(\frac{2 - y}{3} - \frac{1 - y}{2}\right) = 6y\]
\[\iff 2(2 - y) - 3(1 - y) = 6y\]
\[\iff 4 - 2y - 3 + 3y = 6y\]
\[\iff 1 + y = 6y\]
\[\iff 1 = 5y\]
\[\text{so } y = \frac{1}{5}\]
(q) \[ y - \frac{(1 - y)}{2} + 4 = 7 \]

**Solution.** First we transfer all the numbers to the RHS, and then multiply both sides by 2.

\[ y - \frac{(1 - y)}{2} + 4 = 7 \]
\[ \iff y - \frac{(1 - y)}{2} = 3 \]
\[ \iff 2y - (1 - y) = 6 \]
\[ \iff 2y - 1 + y = 6 \]
\[ \iff 3y = 7 \]
\[ \iff y = \frac{7}{3} \]

(r) \[ \frac{y}{3} - (3 - y) = 4 - y \]

**Solution.** Multiplying both sides by 3 gives

\[ y - 3(3 - y) = 3(4 - y) \]
\[ \iff y - 9 + 3y = 12 - 3y \]
\[ \iff 4y = 21 - 3y \]
\[ \iff 7y = 21 \]
\[ \iff y = 3 \]

(s) \[ \frac{d - 1}{2} - \frac{d}{3} = 0 \]

**Solution.** First multiply both sides by LCM(2, 3) = 6.

\[ 6 \left( \frac{d - 1}{2} - \frac{d}{3} \right) = 0 \]
\[ \iff 3(d - 1) - 2d = 0 \]
\[ \iff 3d - 3 - 2d = 0 \]
\[ \iff d = 3 \]

(t) \[ \frac{4 - p}{3} = -8 \]

**Solution.** Multiply both sides by 3:

\[ 4 - p = -24 \]
\[ \iff -p = -28 \]
\[ \iff p = 28 \]

C. Modelling using simple Equations

10. In a triangle the second angle is twice the first and the third angle is three times the first. What are the values of the angles?

**Solution.** Let the first angle be \( x \). Then the second angle is \( 2x \) and the third is \( 3x \). Since the sum of the angles in a triangle is 180°, we have

\[ x + 2x + 3x = 180 \]
\[ 6x = 180 \]
\[ x = 30 \]

Thus the angles are 30°, 60° and 90°.

11. In the following problems, first formulate an appropriate equation and then solve it.

(a) Suppose Mary travels a certain distance on the first day and twice the distance on the next day. If the total distance she travelled is 60 km, how far does she travel on the first day?

**Solution.** Let \( x \) denote the distance travelled on the first day. The distance travelled on the second day is \( 2x \). Since the total distance travelled is 60 km, we must have

\[ x + 2x = 60 \]
\[ 3x = 60 \]
\[ x = 20 \]

so Mary travels 20 km on the first day.

(b) Joy, Pam, Sandra and Lilin each make a donation to the Guide Dogs Association. Sandra gives twice as much as Lilin, Pam gives three times as much as Sandra and Joy gives four times as much as Pam. If their total gift is $132, find the amount of Lilin’s donation, and hence the amount donated by each.

**Solution.** Let the amount given by Lilin be \( x \). Then Sandra gives \( 2x \), Pam gives \( 3 \times 2x = 6x \) and Joy gives \( 4 \times 6x = 24x \). The total donation was $132, so

\[ x + 2x + 6x + 24x = 132 \]
\[ 33x = 132 \]
\[ x = \frac{132}{33} = 4 \]

Thus Lilin gave $4, Sandra gave $8, Pam gave $24 and Joy gave $96. The total of these amounts is $132 as required.
(c) How many litres of a 10% solution of phosphoric acid should be added to 20 litres of a 60% solution to make a 50% solution? All percentages are by volume.

**Solution.** Let \( x \) denote the volume of the 10% solution. Then the volume of the 60% solution is 20, so the volume of the resulting 50% solution is \((x + 20)\). The information is presented in the table below.

<table>
<thead>
<tr>
<th>Vol of solution</th>
<th>10%</th>
<th>60%</th>
<th>50%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vol of Acid</td>
<td>0.1x</td>
<td>0.6(20)</td>
<td>0.5(20 + x)</td>
</tr>
</tbody>
</table>

So we need to solve the equation

\[
0.1x + 0.6(20) = 0.5(20 + x)
\]

\[
\Leftrightarrow 0.1x + 12 = 10 + 0.5x
\]

\[
\Leftrightarrow 2 = 0.4x
\]

\[
\Leftrightarrow x = \frac{2}{0.4} = \frac{20}{4} = 5
\]

We need 5\(\ell\) of the 10% solution and we end up with 25\(\ell\) of the 50% solution.

(d) Two unblended manganese ores contain 40% and 25% manganese respectively by weight. How many tonnes of each must be mixed to give 100 tonnes of blended ore containing 35% manganese?

**Solution.** Let \( x \) be the weight (in tonnes) of the 40% ore.

<table>
<thead>
<tr>
<th>Vol of solution</th>
<th>40%</th>
<th>25%</th>
<th>35%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vol of Acid</td>
<td>0.4x</td>
<td>0.25(100 - x)</td>
<td>0.35(100)</td>
</tr>
</tbody>
</table>

So we need to solve the equation

\[
0.4x + 0.25(100 - x) = 0.35(100)
\]

\[
\Leftrightarrow 0.4x + 25 - 0.25x = 35
\]

\[
\Leftrightarrow 0.15x = 10
\]

\[
\Leftrightarrow x = \frac{10}{0.15} = \frac{1000}{15} = \frac{200}{3}
\]

We need \(\frac{200}{3}\) tonnes of the 40% ore and \(100 - \frac{200}{3} = \frac{300}{3} - \frac{200}{3} = \frac{100}{3}\) tonnes of the 25% ore.

(e) One thousand litres of milk testing 4% butterfat are to be reduced to 3%. How many litres of cream testing 23% butterfat must be separated from the milk to produce the required result? All percentages are by volume.

**Solution.** Let \( x \) denote the volume of cream testing 23% butterfat that is removed. Then the volume of the remaining milk testing 3% butterfat is 1000 - \(x\).

<table>
<thead>
<tr>
<th>Vol of Milk</th>
<th>23%</th>
<th>3%</th>
<th>4%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vol of butterfat</td>
<td>0.23x</td>
<td>0.03(100 - x)</td>
<td>0.04(100)</td>
</tr>
</tbody>
</table>

So we need to solve the equation

\[
0.23x + 0.03(1000 - x) = 0.04(1000)
\]

\[
\Leftrightarrow 0.23x + 30 - 0.03x = 40
\]

\[
\Leftrightarrow 0.2x = 10
\]

\[
\Leftrightarrow x = \frac{10}{0.2} = \frac{100}{2} = 50
\]

We need to remove 50\(\ell\) of cream testing 23% butterfat.

(f) Tank A contains a mixture of 10 litres of water and 5 litres of frother. Tank B contains 12 litres of water and 3 litres of frother. Both tanks have been thoroughly stirred. How many litres should be taken from each tank and combined to obtain an 8 litre solution containing 25% frother by volume? (Frothers are used in mining to float minerals.)

**Solution.** Let \( x \) denote the volume of the mixture from tank A. The fraction of frother in Tank A is \(\frac{5}{17}\). Similarly, the fraction of frother in Tank B is \(\frac{3}{17}\).

<table>
<thead>
<tr>
<th>Vol of solution</th>
<th>23%</th>
<th>3%</th>
<th>4%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vol of Frother</td>
<td>0.23x</td>
<td>0.03(100 - x)</td>
<td>0.04(100)</td>
</tr>
</tbody>
</table>

So we need to solve the equation

\[
0.23x + 0.03(1000 - x) = 0.04(1000)
\]

\[
\Leftrightarrow 0.23x + 30 - 0.03x = 40
\]

\[
\Leftrightarrow 0.2x = 10
\]

\[
\Leftrightarrow x = \frac{10}{0.2} = \frac{100}{2} = 50
\]

We need to remove 50\(\ell\) of cream testing 23% butterfat.
So we need to solve the equation

\[
\frac{5x}{15} + \frac{3}{15}(8 - x) = \frac{1}{4}(8)
\]

\[
\Leftrightarrow 15 \left[ \frac{5x}{15} + \frac{3}{15}(8 - x) \right] = 15 \times 2
\]

\[
\Leftrightarrow 5x + 24 - 3x = 30
\]

\[
\Leftrightarrow 2x = 6
\]

\[
\Leftrightarrow x = 3
\]

So we need 3ℓ of the mixture from Tank A and the remaining 5ℓ from Tank B.

12. **Puzzle** A school has 1,000 students and 1,000 lockers. One day all the students line up outside the school. The first student goes in and opens all the lockers. The second student goes in next and closes all the even numbers lockers. The third student goes in and reverses the state (closes the open ones and opens the closed ones) of all lockers that are multiples of 3. The fourth student goes in and reverses the state of all lockers that are multiples of 4. This continues until the last student goes in and reverses the state of the locker that is a multiple of 1,000.

How many lockers are open at the end, and which ones are they?

**Hint** First consider 10 lockers and solve the problem by hand. Look for a pattern. Justify the pattern and then generalise your solution.

**Solution.** The puzzle may be discussed in lectures. The answer is all the locker numbers that are perfect squares remain open, that is, \(1^2, 2^2, 3^2, \ldots, 31^2 = 961\) remain open, that is, 31 locker in total. Can you justify this answer?