The University of Western Australia

SCHOOL OF MATHEMATICS AND STATISTICS

MATH1050 Calculus C

Tutorial 5, Semester 2 2010

Quadratic Equations

Attempt questions 1 (c), (j), (n), Q2 (a), (m), Q3 (1), (e), (h) before your tutorial class.

A. Solving Quadratic Equations by Factorising

1. Solve the following equations by factorising.

(a) \( x^3 - 3x = 0 \), \( x = 0, 3 \)
(b) \( x^2 - 5x + 4 = 0 \), \( x = 4, 1 \)
(c) \( x^2 - 5x - 14 = 0 \), \( x = 7, -2 \)
(d) \( x^2 - x - 42 = 0 \), \( x = 7, -6 \)
(e) \( 3x^2 - 6x = -3 \), \( x = -1 \)
(f) \( 3x^2 - 7x + 2 = 0 \), \( x = 1/3, 2 \)
(g) \( 12x^2 = 8x + 15 \), \( x = 3/2, -5/6 \)
(h) \( (x - 2)(x + 1) = 4 \), \( x = 3, -2 \)
(i) \( f(t + 4) - 1 = 4 \), \( t = -5, 1 \)
(j) \( 2x(x - 2) = x + 3 \), \( x = -1/2, 3 \)
(k) \( (x + 2)^2 = 4x \) No real sol.
(l) \( x^4 + x^2 - 6 = 0 \), \( x = \pm \sqrt{2} \)
(m) \( 2x^2 + 3x - 14 = 0 \), \( x = -7/2, 2 \)
(n) \( x + \frac{15}{x} = 8 \), \( x = 5, 3 \)
(o) \( 3x - \frac{4}{x} = 11 \), \( x = -1/3, 4 \)
(p) \( x = \frac{5}{x + 4} \), \( x = 5, -1 \)
(q) \( (x + 2)(x + 3) = 12 \), \( x = -6, 1 \)
(r) \( x = \frac{5}{x + 4} \), \( x = -5, 1 \)
(s) \( (x + 14)^2 + x^2 = 34^2 \), \( x = -30, 16 \)
(t) \( x^2 + (2x + 4)^2 = (2x + 6)^2 \), \( x = 10, -2 \)

B. Solving Quadratic Equations by using the Quadratic Formula

2. Solve the following equations by using the quadratic formula.

(a) \( x^2 + 4x + 1 = 0 \), \( x = -2 \pm \sqrt{3} \)
(b) \( 3x^2 + 7x + 2 = 0 \), \( x = -1/3, 2 \)
(c) \( 2x^2 + 3x + 1 = 0 \), \( x = -1/2, -1 \)
(d) \( x^2 + 3x + 4 = 0 \) No real sol.
(e) \( \frac{3}{2} x^2 + 4x + 1 = 0 \), \( x = (-4 \pm \sqrt{10})/3 \)
(f) \( x^2 + x - 1 = 0 \), \( x = (-1 \pm \sqrt{5})/2 \)
(g) \( 7x^2 + x - 2 = 0 \), \( x = (-1 \pm \sqrt{25})/14 \)
(h) \( 4x^2 + 4x + 3 = 0 \), No real sol.
(i) \( 4x^2 - 3x - 4 = 0 \) \( x = (3 \pm \sqrt{73})/8 \)
(j) \( x^2 + 4x - 6 = 0 \), \( x = -2 \pm \sqrt{10} \)
(k) \( 2x^2 - 4x - 3 = 0 \) \( x = (2 \pm \sqrt{10})/2 \)
(l) \( x(x - 2) = 4 \), \( x = 1 \pm \sqrt{5} \)
(m) \( x^2 + 2x = 1 \), \( x = -1 \pm \sqrt{2} \)

C. Number of solutions

3. Determine the number of solutions for the following equations without solving them.

(a) \( x^2 + 3x + 1 = 0 \) Two real sol.  (d) \( x^2 + 3x + 6 = 0 \) No real sol.  (g) \( 7x^2 + x - 2 = 0 \) Two real sol.
(b) \( 3x^2 + 4x + 2 = 0 \) No real sol.  (e) \( \frac{3}{2} x^2 + 4x + 6 = 0 \) No real sol.  (h) \( 4x^2 + 3x + 1 = 0 \) No real sol.
(c) \( 2x^2 + 3x + 5 = 0 \) No real sol.  (f) \( x^2 + 5x - 1 = 0 \) Two real sol.  (i) \( 4x^2 - 5x + 4 = 0 \) No real sol.

D. Modelling using Quadratic Equations

4. One positive number exceeds three times another positive number by 5. The product of the numbers is 68. Find the numbers. Sol.: The numbers are 4 and 17.
5. The hypotenuse of a right angled triangle is 34cm. Of the remaining two sides, one is 14cm longer than the other. Find the unknown sides. Sol.: 16cm and 30cm.

6. A group of zoologists was studying the effect on the body weight of rats of varying the amount of yeast in their diet. By changing the percentage $P$ of yeast in the diet, the average weight gain, $G$ (in grams), over time was estimated to be $G = -200P^2 + 200P + 20$. What percentage of yeast would you expect to give an average weight gain of 70 grams? Sol.: 1/2%.

7. A small oil field contains 20 wells and each produces 200 barrels of oil daily. A recent report estimates that for each additional well drilled, daily production at each well would decrease by five barrels. If the oil company wishes to increase oil production to 4420 barrels a day, how many new wells should be drilled? Sol.: 6 more.

8. The sum of the squares of two consecutive even real numbers is 52. Find the numbers. Sol.: Either -6 and -4 or 4 and 6.

9. The length and breadth of a rectangle are $(x + 4)$ cm and $x$ cm respectively.
   (a) Write the expression for
      i. the perimeter of the rectangle,
      ii. the length of the side of a square with the same perimeter.
   (b) If the sum of the areas of the square and the rectangle is 94cm$^2$, find $x$.
   Sol.: $x = 5$

10. A small swimming pool can be filled by two pipes in 3 hours. If the pipes are used separately, it takes 12 hours for the larger pipe used alone and 8 hours for the smaller pipe used alone to fill the pool. Find the time in which it will be filled by each pipe singly. Sol.: Smaller 12hrs and larger 4 hrs.

11. An object is launched at 19.6 meters per second (m/s) from a 58.8-meter tall platform. The equation for the object’s height $s$ at time $t$ seconds after launch is $s(t) = -4.9t^2 + 19.6t + 58.8$, where $s$ is in meters. When does the object strike the ground? Sol.: After 6 secs.

12. A framer at a photo gallery wants to frame a print with a matte of uniform width all around the print. To make it pleasing to the eye, the area of the matte should equal the area of the print. If the print measures 40 cm by 60 cm, how wide should the matte be? Sol.: 10cm.

13. A machine produces open boxes using square sheets of plastic. The machine cuts equal sized squares measuring 2cm on a side from each corner of the sheet, and then shapes the plastic into an open box by turning up the sides. If each box must have a volume of 242 cubic cm, find the length of the sides of the open box. Sol.: 11cm.

14. A model rocket is shot into the air and its path is approximated by $h = -5t^2 + 30t$ where $h$ is the height of the rocket above the ground in metres and $t$ is the elapsed time in seconds. When will the rocket hit the ground? Sol.: 6 secs.

15. A motor boat travels 400km. If the boat went 18km/h faster, it could have travelled 600 km in the same amount of time. What was the original speed of the boat? Sol.: 36km/h

16. A parallelogram has a base of length $2x + 1$ and a height of $x + 3$ and has an area of 42 square units. Find the base and height of the parallelogram. (A=BH) Sol.: $x = 3$.

17. Ahmed has first half of a treasure map, which indicates that the treasure is buried in the desert $(2x + 6)$ paces from Castle Rock. Vanessa has the other half of the map. Her half indicates that to find the treasure, one must go to the Castle Rock, walk $x$ paces to the East, More Difficult Factorisation, 1.7 Simplifications /LIj, North, and then walk $(2x + 4)$ paces to the East. If they share their information, then they can find $x$ and save a lot of digging. What is $x$? Sol.: $x = 10$ paces

18. What value of $c$ makes $x^2 - x + c$ a perfect square? Sol.: $c = 1/4$.

19. The path of the ball for many golf shots can be modelled by a quadratic function. The path of a golf ball hit at an angle of about 10 degrees to the horizontal can be modeled by the function, $h = -0.002d^2 + 0.4d$, where $h$ is the height of the ball, in meters, and $d$ is the horizontal distance the ball travels, in meters, until it first hits the ground.
   (a) What is the maximum height reached by the ball? Sol.: 20m. b) What is the horizontal distance of the ball from the golfer when the ball reaches its maximum height? Sol.: 100m.
   (b) What distance does the ball travel horizontally until it first hits the ground? Sol.: 200m.
20. A rectangle corral is to be built using 70m of fencing. If the fencing has to enclose all four sides of the corral, what is the maximum possible area of the corral in square metres? Sol.: 306.25 m^2

21. Frank’s Specialty Chocolates makes a house blend from two types of beans, one selling for $9.05 per kg, and the other selling for $6.25 per kg. His house blend sells for $7.37 per kg. If he is using 9 kg of the 6.25/kg beans, how many kilogrammes of the $9.05/kg beans does he need to make his house blend? Sol.: 6kg.
Let the second number be \( x \). Then the first number is \( 3x + 5 \). Further,
\[
(3x + 5) = 68
\]
\[
3x^2 + 5x - 68 = 0
\]
\[
(3x + 17)(x - 4) = 0
\]
\[
x = 4 \text{ or } -\frac{17}{3}
\]

Since \( x \) is positive, we must have \( x = 4 \). Thus the numbers are \( 4, 3(4) + 5 = 17 \).

By Pythagoras' theorem,
\[
(\sqrt{x+14})^2 + x^2 = 34^2
\]
\[
x + 14 \Rightarrow 2x^2 + 28x - 960 = 0
\]
\[
x^2 + 14x - 480 = 0
\]
\[
(x + 30)(x - 16) = 0
\]
\[
x = -30 \text{ or } 16.
\]

Since \( x \) is positive, we must have \( x = 16 \). Thus the remaining sides are 16 m and 30 cm.
6. We need to solve the equation

\[-200P^2 + 200P + 20 = 70\]

\[-200P^2 + 200P - 50 = 0\]

\[(\div 50)\]

\[4P^2 - 4P + 1 = 0\]

\[(2P - 1)^2 = 0\]

\[P = \frac{1}{2}\]

Thus \(\frac{1}{2}\%\) of yeast in the diet would give an expected weight gain of 70g.

7. Let \(W\) be the number of new wells drilled.

Current production is 200 barrels per well. With the new wells, daily production per well is 200 - 5W (i.e., reduction of 5 barrels per day per well).

Total number of wells is 20 + W. Thus

Total daily production = (# of wells) × (daily production per well)

\[
(200 - 5W)(20 + W) = 4420
\]

\[4000 + 200W - 100W - 5W^2 = 4420\]

\[5W^2 - 100W + 420 = 0\]

\[W^2 - 20W + 84 = 0\]

\[(W - 14)(W - 6) = 0\]

\[W = 6, 14\]

Thus we need to drill 6 more wells.
8. Let the first number = \( x \).
The second number = \( x + 2 \).

\[
x^2 + (x+2)^2 = 52
\]
\[
\iff x^2 + x^2 + 4x + 4 = 52
\]
\[
\iff 2x^2 + 4x - 48 = 0
\]
\[
\iff x^2 + 2x - 24 = 0
\]
\[
\iff (x+6)(x-4) = 0
\]
\[
\therefore x = -6 \text{ or } x = 4
\]

The numbers are -6 and -4, or 4 and 6.

9.

(a) \[
\begin{array}{c}
\frac{x}{\text{cm}} \\
\frac{(x+4)}{\text{cm}}
\end{array}
\]

(i) Perimeter = \( 2(x+4) + 2x \)
\[
= 2x + 8 + 2x
\]
\[
= (4x + 8) \text{ cm}
\]

(ii) A square with perimeter \( 4x + 8 \) has side length \( \frac{4x+8}{4} = (x+2) \text{ cm} \)

(b) Area of the rectangle = \( x(x+4) \)
Area of the square = \( (x+2)^2 \)

\[
x(x+4) + (x+2)^2 = 94
\]
\[
\iff x^2 + 4x + x^2 + 4x + 4 = 94
\]
\[
\iff 2x^2 + 8x - 90 = 0
\]
\[
\iff x^2 + 4x - 45 = 0
\]
\[
\iff (x+9)(x-5) = 0
\]
\[
\therefore x = 5 \text{ cm, since negative soln not allowed}
\]
Q10.

Let the time the smaller pipe takes = \( \frac{1}{x} \) hrs singly
· The larger pipe takes \( (x-8) \) hrs singly

In 1 hr the small pipe has filled \( \frac{1}{x} \) of the pool
and the larger pipe \( \frac{1}{x-8} \)

Thus in 3hrs:
\[
\frac{3}{x} + \frac{3}{x-8} = 1 \quad \Rightarrow \quad 1 \text{ whole pool}
\]

\[
\Leftrightarrow \quad \frac{3(x-8) + 3x}{x(x-8)} = 1
\]
\[
\Leftrightarrow \quad 3(x-8) + 3x = x(x-8)
\]
\[
\Leftrightarrow \quad 3x - 24 + 3x = x^2 - 8x
\]
\[
\Leftrightarrow \quad x^2 - 14x + 24 = 0 \quad \Rightarrow \text{rearranging}
\]
\[
\Leftrightarrow \quad (x-12)(x-2) = 0
\]
\[ x = 12 \quad \text{since } x = 2 \text{ not allowed} \]

Thus the smaller pipe takes 12 hrs singly and
the larger pipe 4 hrs.

Q11.

When object strikes the ground \( s = 0 \)
Thus we are solving \( -4.9t^2 + 19.6t + 58.8 = 0 \) \( \div -4.9 \)
\[
\Leftrightarrow \quad t^2 - 4t - 12 = 0 \quad \Leftrightarrow \quad (t+2)(t-6) = 0
\]
\[ t = 6 \quad \text{since neg soln not allowed} \]
· The object strikes the ground after 6 secs.
Area of the print = 40 \times 60 = 2400 \text{ cm}^2

Area of the matte = (40+2x)(60+2x) - 2400

\Rightarrow (40+2x)(60+2x) - 2400 = 2400

\Rightarrow 2400 + 80x + 1200x + 4x^2 - 2400 = 2400

\Rightarrow 4x^2 + 200x - 2400 = 0

\Rightarrow x^2 + 50x - 600 = 0

\Rightarrow (x-10)(x+60) = 0

\therefore x = 10 \text{ cm, since neg soln not allowed.}

Vol = x \times x \times 2 = 2x^2

2x^2 = 242

x^2 = 121

x = 11 \text{ cm}

h = -5t^2 + 30t = 0

\Rightarrow -5t(t-6) = 0

\therefore t = 6 \text{ secs } \quad \text{(t=0 is when it is still on the ground)}
Q15. \[ \text{Speed} = \frac{\text{distance}}{\text{time}} \quad \text{time} = \frac{\text{distance}}{\text{speed}} \]

Let the speed of the boat = \( x \)

\[ \frac{400}{x} = \frac{600}{x+18} \]

\[ \Rightarrow 600x = 400(x+18) \]

\[ \Rightarrow 600x = 400x + 7200 \]

\[ \Rightarrow 200x = 7200 \]

\[ \Rightarrow x = \frac{7200}{200} = 36 \text{ km/hr} \]

Q16.

\[ A = BH \]

\[ (2x+1)(x+3) = 42 \]

\[ \Rightarrow 2x^2 + 6x + x + 3 = 42 \]

\[ \Rightarrow 2x^2 + 7x - 39 = 0 \]

\[ \Rightarrow (2x+13)(x-3) = 0 \]

\[ x = 3 \quad \text{since neg soln not allowed} \]

Q17.

Using Pythagoras:

\[ a^2 + b^2 = c^2 \]

\[ \Rightarrow x^2 + (2x+4)^2 = (2x+6)^2 \]

\[ \Rightarrow x^2 + 4x^2 + 16x + 16 = 4x^2 + 24x + 36 \]

\[ \Rightarrow x^2 - 8x - 20 = 0 \]

\[ \Rightarrow (x-10)(x+2) = 0 \]

\[ \therefore x = 10 \text{ paces} \]

Q18.

\[ x^2 - x + c \]

\[ c = \left( -\frac{1}{2} \right)^2 = \frac{1}{4} \]
Q19

\[ h = -0.002d^2 + 0.4d \]

\[ 20 \]

\[ 100 \quad 200 \rightarrow d \]

a) To find turning point: \[ 2 \times 0.5 \quad d = \frac{-0.4}{2(-0.002)} = 100 \]

Substitute \( d = 100 \) : \[ h = -0.002(100)^2 + 0.4(100) \]
\[ h = 20 \text{m (max height)} \]

b) From turning point \( d = 100 \text{m} \)

c) Line of symmetry half way between the 'roots' of the function, so \( d = 200 \text{m} \) (see above)

Q20

\[ \text{Area} = x(35-x) = 35x - x^2 \]

This graph has turning point \( (17.5, 306.25) \) thus the max area is \( 306.25 \text{m}^2 \)

Q21

The number of \( \text{Kg} \) of \$9.05/kg = \( x \)

Total cost = quantity \times \text{price}

\[ 9.05x + 9(6.25) = 7.37(x+9) \]

\[ 9.05x + 56.25 = 7.37x + 66.33 \]

\[ 1.68x = 10.08 \]

\[ \therefore \quad x = 6 \text{ Kg} \]