3A7 Dynamical Systems and Chaos

(MATH3327)

1. Foundations
Mathematical Models

- **Maps:** \( x_{t+1} = f(x_t), x_t \in S \subseteq \mathbb{R}^n, t \in \mathbb{Z} \)
  - \( S \subseteq \mathbb{R}^n \) is state space
  - \( t \in \mathbb{Z}, \) time is discrete

- **Flows:** \( \frac{dx}{dt} = F(x, t), x \in \mathbb{R}^n, t \in \mathbb{R} \)
  - \( \mathbb{R}^n \) is state space, \( t \in \mathbb{R} \) time is continuous
  - Flows are ordinary differential equations (ODE)
  - Autonomous flow \( \dot{x} = F(x), x \in \mathbb{R}^n, \)
    where \( F \) does not depend on \( t \)
  - Convenient notation, \( \dot{x} = \frac{dx}{dt} \)

- Partial differential equations (PDE)
Autonomous Flows

• What does $\dot{x} = F(x)$, $x \in \mathbb{R}^n$, mean?
• $F(x)$ is a vector field, $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$
  • $\mathbb{R}^n$ is state, $\mathbb{R}^n$ is a tangent vector

Solution $p(t)$ of $\dot{x} = F(x)$
• Path $p(t)$, $p : \mathbb{R} \rightarrow \mathbb{R}^n$
• Tangent vector of $p(t)$ at $x$ is $F(x)$,
• that is $\frac{d}{dt}p(t) = F(p(t))$
• $p(t)$ is not necessarily defined for all $t \in \mathbb{R}$

Trajectories $q(t, x)$, $q : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$
• $q(0, x) = x$ initial condition
• $\frac{d}{dt}q(t, x) = F(q(t, x))$
• $q(t + s, x) = q(s, q(t, x))$ semigroup property

Evolution operator $\Phi_t x = q(t, x)$
• $\Phi_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$
• $\Phi_0$ is the identity map, $\Phi_0(x) = x$
• $\Phi_{t+s} = \Phi_s \circ \Phi_t$,
Existence and Uniqueness

If $F: \mathbb{R}^n \to \mathbb{R}^n$ is a differentiable vector field, then for each $x_0 \in \mathbb{R}^n$ there is a $T > 0$ such that $\dot{x} = F(x)$ has a unique solution $q(t, x_0)$ for $-T < t < T$ where $q(0, x_0) = x_0$.

—⋄—

- If $F: \mathbb{R}^n \to \mathbb{R}^n$ is a differentiable vector field,
  - that is, all partial derivatives $\frac{\partial F_i}{\partial x_j}$ exist everywhere
- then for each $x_0 \in \mathbb{R}^n$ (initial condition)
- there is a $T > 0$
- such that $\dot{x} = F(x)$ has a unique solution $q(t, x_0)$
- for $-T < t < T$
- where $q(0, x_0) = x_0$. 
Consequences of Existence and Uniqueness Theorem

For dynamical systems defined by differentiable vector fields:

- Two different states cannot evolve to the same state at the same time
- If two different states do evolve to the same state, then they are on the same trajectory
- Trajectories never intersect, cross, nor merge
Continuity of Solutions

If $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a differentiable vector field, then for each $x_0 \in \mathbb{R}^n$ there is an $\epsilon > 0$ and $T > 0$ such that $\dot{x} = F(x)$ has a solutions $q(t, x)$ for $-T < t < T$ and $\|x - x_0\| < \epsilon$ where $q(0, x) = x$, and $q(t, x)$ is continuous in $t$ and $x$. 

If $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a differentiable vector field, then for each $x_0 \in \mathbb{R}^n$

- there is an $\epsilon > 0$ and $T > 0$
- such that $\dot{x} = F(x)$ has solutions $q(t, x)$
- for $-T < t < T$ and $\|x - x_0\| < \epsilon$
- where $q(0, x) = x$,
- and $q(t, x)$ is continuous in $t$ and $x$. 
Consequences
of Continuity of Solutions Theorem

The neighbourhood of any state
- evolves as one piece without division
- deforms continuously
Examples of $\dot{x} = F(x)$

1. $\dot{x} = a + bx$, $x \in \mathbb{R}$
2. $\dot{x} = Ax$, $x \in \mathbb{R}^n$, $A$ is a matrix
3. $x \in \mathbb{R}$ and
   \[ F(x) = \begin{cases} 
   +1, & x \leq 0 \\
   -1, & x > 0 
   \end{cases} \]
4. $x \in \mathbb{R}$ and
   \[ F(x) = \begin{cases} 
   +1, & x < 0 \\
   0, & x = 0 \\
   -1, & x > 0 
   \end{cases} \]
5. $x \in \mathbb{R}$ and
   \[ F(x) = \begin{cases} 
   \sqrt{x}, & x > 0 \\
   0, & x \leq 0 
   \end{cases} \]
6. $\dot{x} = x^2$, $x \in \mathbb{R}$
For $\dot{x} = F(x)$, $x \in \mathbb{R}$, there are three useful graphs:
- The graph of $F(x)$. (The “vector field” is just a number.)
- Graphs of state $x$ versus time $t$ for different trajectories
- State space diagram, showing fixed points and flow

Discontinuous $F$ may lead to states where $\dot{x} = F(x)$ has no possible solution

Continuous but not differentiable $F$ can lead to non-unique trajectories

One dimensional flows are characterized by the number of fixed points and the direction of flow between them.
- Often this geometric characterization is sufficient
- What about higher dimensional systems?
- What is the equivalence class of a dynamical systems?
Consider two dynamical systems:

- $\dot{x} = F(x), \; x \in \mathbb{R}^n$, has solutions $q(t, x)$
- $\dot{y} = G(y), \; y \in \mathbb{R}^n$, has solutions $r(t, y)$

These are called equivalent on a domain $U \subseteq \mathbb{R}^n$ if

- there exists a change of coordinates $y = \Psi(x)$
  - that is, a one-to-one mapping $\Psi: U \rightarrow \mathbb{R}^n$,
- such that $r(t, \Psi(x)) = \Psi(q(t, x))$
- for all $x$ and $t$ where $q(t, x)$ exists in $U$.

Equivalence is also called topological conjugacy.
Fundamental Theorem of Flows

If $F: \mathbb{R}^n \to \mathbb{R}^n$ is a differentiable vector field, $x_0 \in \mathbb{R}^n$, $\epsilon > 0$, $F(x) \neq 0$ for $\|x - x_0\| < \epsilon$, $v \in \mathbb{R}^n$, $v \neq 0$, then there exists a differentiable change of coordinates $y = \Psi(x)$ for $\|x - x_0\| < \epsilon$ such that $\dot{x} = F(x)$ is equivalent to $\dot{y} = v$.

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- If $F: \mathbb{R}^n \to \mathbb{R}^n$ is a differentiable vector field,
- $x_0 \in \mathbb{R}^n$, $\epsilon > 0$,
- $F(x) \neq 0$ for $\|x - x_0\| < \epsilon$,
- $v \in \mathbb{R}^n$, $v \neq 0$,
- then there exists a differentiable change of coordinates $y = \Psi(x)$ (\(\Psi : U \to \mathbb{R}^n\)) for $\|x - x_0\| < \epsilon$ (\(U = \{x \in \mathbb{R}^n : \|x - x_0\| < \epsilon\}\)) such that $\dot{x} = F(x)$ is equivalent to $\dot{y} = v$. 
Trajectories of $\dot{y} = v$ are $r(t, y_0) = y_0 + vt$

If $x_0$ is not a fixed point,
- then the flow near $x_0$
- is equivalent to parallel trajectories
- moving at constant speed,

The only interesting local behaviour occurs close to fixed points

Assume from now on vector fields are differentiable.
Consider two equivalent dynamical systems:

- $\dot{x} = F(x), \ x \in U \subseteq \mathbb{R}^n$, has solutions $q(t, x)$
- $\dot{y} = G(y), \ y \in V \subseteq \mathbb{R}^n$, has solutions $r(t, y)$
- \textit{Differentiable} change of coordinates $y = \Psi(x)$
- $\Psi: U \rightarrow V$ one-to-one, onto, differentiable

\[
\begin{align*}
y &= \Psi(x) & \Psi: U \rightarrow V, \text{ one-to-one, onto} \\
\dot{y} &= d\Psi(x)\dot{x} & d\Psi(x) \text{ is Jacobian evaluated at } x \\
 &= d\Psi(x)F(x) \end{align*}
\]

$\therefore G(y) = d\Psi(\Psi^{-1}(y))F(\Psi^{-1}(y))$

because $x = \Psi^{-1}(y)$ \quad $\Psi^{-1}: V \rightarrow U$

Alternatively $\dot{x} = d\Psi^{-1}(y)G(y) = d\Psi^{-1}(\Psi(x))G(\Psi(x))$

or $F(x) = d\Psi(x)^{-1}G(\Psi(x))$

since $d\Psi(x)^{-1} = d\Psi^{-1}(\Psi(x))$