3A7 Dynamical Systems and Chaos

(MATH3327)

2. Fixed Points
A state $x^*$ where $F(x^*) = 0$ is a fixed point.

It remains fixed under the flow or evolution operator.

The fixed point $x^*$ is a entire trajectory, defined for all time.

The uniqueness theorem implies a trajectory cannot arrive at, or leave from, a fixed point state in finite time.

Other trajectories can

- approach a fixed point
- either forward or backward in time
- but only asymptotically
Linearization Theorem

If $F: \mathbb{R}^n \to \mathbb{R}^n$ is a differentiable vector field, and $x^*$ is a fixed point such that the Jacobian matrix $A = dF(x^*)$ has only eigenvalues with non-zero real parts, then there exists $\epsilon > 0$ such that for $\|x - x^*\| < \epsilon$ the dynamics of $\dot{x} = F(x)$ is equivalent to the linear system $\dot{y} = Ay$. 
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- and $x^*$ is a fixed point
- such that the Jacobian matrix $A = dF(x^*)$
- has only eigenvalues with non-zero real parts,
- then there exists $\epsilon > 0$ such that
- for $\|x - x^*\| < \epsilon$ ($U = \{x \in \mathbb{R}^n : \|x - x^*\| < \epsilon\}$)
- the dynamics of $\dot{x} = F(x)$ is equivalent to
- the linear system $\dot{y} = Ay$. ($G(y) = Ay$)
Consequences of Linearization Theorem

- Fixed point whose eigenvalues have non-zero real parts are called hyperbolic fixed points
- Complements the Fundamental Theorem of Flows
- Behaviour near fixed point is equivalent to a linear system
- Behaviour determined entirely by Jacobian matrix, at evaluated at the fixed point $x^*$

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- What do linear systems $\dot{y} = Ay$ look like?
  - Make another linear change of coordinates $z = Py$
  - to obtain equivalent system $\dot{z} = Jz$
  - where $A = P^{-1}JP$ (Recall slide 23 of previous section.)
Jordan Form Theorem

Recall from linear algebra

For any square matrix $A$ there exists an invertible matrix $V$ and block diagonal matrix $J$ such that $A = VJV^{-1}$. The blocks of the matrix $J$ are of the form

$$
\begin{pmatrix}
\lambda & 1 & 0 \\
& \ddots & \ddots \\
& & \lambda & 1 \\
0 & & & \lambda
\end{pmatrix}
$$

where $\lambda$ is an eigenvalue of $A$.

- The total number of appearances of $\lambda$ in $J$ is the eigenvalue’s *algebraic multiplicity*, that is, multiplicity of the root in the characteristic polynomial $\det(A - \lambda I) = 0$.
- The total number of blocks containing $\lambda$ is its *geometric multiplicity*, that is, the dimension of $\ker(A - \lambda I)$. 

Jordan Forms for $2 \times 2$ complex matrices

$$J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

Three forms correspond to three cases:

- Distinct eigenvalues
- Algebraic multiplicity 2, geometric multiplicity 2
- Algebraic multiplicity 2, geometric multiplicity 1
Jordan Forms for $2 \times 2$ real matrices

- \[
\begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix}, \text{ distinct real eigenvalues } \lambda_1, \lambda_2 \in \mathbb{R}
\]

- \[
\begin{pmatrix}
\lambda & 0 \\
0 & \lambda
\end{pmatrix}, \text{ one real eigenvalue } \lambda \text{ with distinct eigenvectors}
\]

- \[
\begin{pmatrix}
\lambda & 1 \\
0 & \lambda
\end{pmatrix}, \text{ one real eigenvalue } \lambda \text{ with only one eigenvector}
\]

- \[
\begin{pmatrix}
\alpha + i\beta & 0 \\
0 & \alpha - i\beta
\end{pmatrix}, \text{ complex conjugate eigenvalues}
\]

- \[
\begin{pmatrix}
\alpha & \beta \\
-\beta & \alpha
\end{pmatrix}, \text{ alternative form for complex conjugate } \alpha \pm \beta i
Flows near hyperbolic fixed points in $\mathbb{R}^2$

Flows in $\mathbb{R}^2$ near hyperbolic fixed points are equivalent to linear flows

$$
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = A \begin{pmatrix} x \\
y \end{pmatrix}
$$

But these are equivalent to the linear flows

$$
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = J \begin{pmatrix} x \\
y \end{pmatrix}
$$

where $J$ is a $2 \times 2$ Jordan form matrix.

Because $A = VJV^{-1}$ the equivalence is achieved through linear change of coordinates $\Psi(x, y) = V^{-1} \begin{pmatrix} x \\
y \end{pmatrix}$
Fixed points: Nodes and Saddles

\[ J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \]

- distinct real eigenvalues \( \lambda_1, \lambda_2 \in \mathbb{R} \)
- \( \dot{z} = Jz \) has trajectories

\[ q(t, (z_1, z_2)) = (z_1 e^{\lambda_1 t}, z_2 e^{\lambda_2 t}) \]

There are three distinct hyperbolic flows:

- \( \lambda_1 < \lambda_2 < 0 \): stable node
- \( 0 < \lambda_1 < \lambda_2 \): unstable node
- \( \lambda_1 < 0 < \lambda_2 \): saddle

What happens when one or both eigenvalues are zero?
Fixed points: Focii

- \( J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \)
- Complex conjugate eigenvalues \( \lambda = \alpha \pm \beta i, \alpha, \beta \in \mathbb{R} \)
- \( \dot{z} = Jz \) has trajectories of the form

\[
q(t, (z_1, z_2)) = (\rho e^{\alpha t} \cos(\beta t + \phi), \rho e^{\alpha t} \sin(\beta t + \phi))
\]

where \( \rho = \sqrt{z_1^2 + z_2^2} \) and \( \tan \phi = z_2/z_1 \)

There are three distinct flows:
- \( \alpha < 0 \): stable focus
- \( \alpha > 0 \): unstable focus
- \( \alpha = 0 \): centre (non-hyperbolic, has closed orbits)
Fixed points: Degenerate nodes

- $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$
- repeated eigenvalue $\lambda \in \mathbb{R}$
- $\dot{z} = Jz$ has trajectories

$q(t, (z_1, z_2)) = ?$

There are two distinct *hyperbolic* flows:
- $\lambda < 0$: stable degenerate node
- $\lambda > 0$: unstable degenerate node

What happens when $\lambda = 0$ zero?
Role of Taylor’s Theorem

- Recall Taylor’s Theorem for $f: \mathbb{R} \to \mathbb{R}$,

\[ f(x + h) = f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \text{higher order terms} \]
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