6. Bifurcation Theory
Consider a dynamical system $\dot{x} = F(x, \mu)$, $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}^k$, where $F$ is differentiable in both $x$ and $\mu$

Important questions:

- What $\mu$ give structurally stable systems?
- What are the structurally stable systems?
- How do systems change at bifurcations?
- What kind of bifurcations are possible?

Consider then

- 1D systems: $\dot{x} = f(x, \mu)$, $x \in \mathbb{R}$, $\mu \in \mathbb{R}$
- 2D systems: $x, y, \mu \in \mathbb{R}$

\[
\begin{align*}
\dot{x} & = f(x, \mu) \\
\dot{y} & = -y
\end{align*}
\]
Let $\dot{x} = \mu - x^2$, $x \in \mathbb{R}$, $\mu \in \mathbb{R}$

Fixed points:

$$x^* = \begin{cases} 
\pm \sqrt{\mu} & \mu > 0 \\
0 & \mu = 0 \\
\text{none} & \mu < 0 
\end{cases}$$

Stability: $f'(x) = -2x$

$$f'(\pm \sqrt{\mu}) = -2\sqrt{\mu} < 0 \quad \text{stable } \mu > 0$$

$$f'(-\sqrt{\mu}) = +2\sqrt{\mu} > 0 \quad \text{unstable } \mu > 0$$

$$f'(0) = 0 \quad \text{non-hyperbolic } \mu = 0$$

Flows

Bifurcation diagram
Let $\dot{x} = \mu x - x^2$, $x \in \mathbb{R}$, $\mu \in \mathbb{R}$

Fixed points: $x^* = 0, \mu$

Stability: $f'(x) = \mu - 2x$

- $f'(\mu) = -\mu$ stable $\mu > 0$, unstable $\mu < 0$
- $f'(0) = \mu$ unstable $\mu > 0$, stable $\mu < 0$
- both non-hyperbolic when $\mu = 0$

Flows

Bifurcation diagram
Supercritical Pitchfork Bifurcation

- Let \( \dot{x} = \mu x - x^3, \ x \in \mathbb{R}, \ \mu \in \mathbb{R} \)
- Fixed points:
  \[
  x^* = \begin{cases} 
    0, \pm \sqrt{\mu} & \mu > 0 \\
    0 & \mu \leq 0 
  \end{cases}
  \]
- Stability: \( f'(x) = \mu - 3x^2 \)
  \[
  f'(\pm \sqrt{\mu}) = -2\mu < 0 \quad \text{stable } \mu > 0
  \]
  \[
  f'(0) = \mu \quad \text{stable } \mu < 0, \text{ unstable } \mu > 0
  \]
  \[
  f'(0) = 0 \quad \text{non-hyperbolic } \mu = 0
  \]
- Flows
- Bifurcation diagram
Let $\dot{x} = \mu x + x^3$, $x \in \mathbb{R}$, $\mu \in \mathbb{R}$

- **Fixed points:**
  \[ x^* = \begin{cases} 
  0, \pm \sqrt{|\mu|} & \mu < 0 \\
  0 & \mu \geq 0 
  \end{cases} \]

- **Stability:**
  \[ f'(x) = \mu + 3x^2 \]
  \[ f'(\pm \sqrt{|\mu|}) = 2|\mu| > 0 \quad \text{unstable } \mu < 0 \]
  \[ f'(0) = \mu \quad \text{stable } \mu < 0, \text{ unstable } \mu > 0 \]
  \[ f'(0) = 0 \quad \text{non-hyperbolic } \mu = 0 \]

- **Flows**
- **Bifurcation diagram**
Example

- Pendulum with torque $\tau$
  \[
  \dot{\theta} = \omega \\
  \dot{\omega} = -\frac{g}{l} \sin \theta - \delta \omega + \tau, \quad \delta > 0
  \]

- Fixed points: $\omega = 0$ and $(g/l) \sin \theta = \tau$
  \[
  \theta^* = \begin{cases} 
  \theta_1 = \arcsin(g/l), \theta_2 = \pi - \theta_1 & |\tau| < g/l \\
  \pi/2 & |\tau| = g/l \\
  \text{none} & |\tau| > g/l
  \end{cases}
  \]

- Stability
- Bifurcation diagrams
Consider a system of \((x, y) \in \mathbb{R}^2\) defined by circular-polar coordinates (i.e. \(x = r \cos \theta\) and \(y = r \sin \theta\))

\[
\begin{align*}
\dot{r} &= \alpha r - r^3 \\
\dot{\theta} &= \omega 
\end{align*}
\]

\((\omega \neq 0\) is a constant)

In terms of \(r\) this looks like the non-negative part a supercritical pitchfork bifurcation

Fixed point \(r^* > 0\) is a periodic orbit in \((x, y)\)–plane

Bifurcation creates of periodic orbit around a fixed point

Two possible bifurcations:

- \textit{supercritical} giving a stable periodic orbit
- \textit{subcritical} giving an unstable periodic orbit, that is, when \(\dot{r} = \alpha r + r^3\)
Hopf Bifurcation

- Given that \( x = r \cos \theta \) and \( y = r \sin \theta \), the system

\[
\begin{align*}
\dot{r} &= \alpha r - r^3 \\
\dot{\theta} &= \omega
\end{align*}
\]

- In terms of \( x \) and \( y \) is

\[
\begin{align*}
\dot{x} &= \alpha x - \omega y - x^3 - xy^2 \\
\dot{y} &= \omega x + \alpha y - x^2 y - y^3
\end{align*}
\]

- Jacobian at \((x, y) = (0, 0)\)

\[
DF(0, 0) = \begin{pmatrix} \alpha & -\omega \\ \omega & \alpha \end{pmatrix}
\]

has eigenvalues \( \alpha \pm i\omega \)

- Bifurcation occurs when focus changes stability
Hopf Bifurcation Theorem

Creation of periodic orbit from a focus

- Let \( \dot{x} = f(x, y, \mu) \) and \( \dot{y} = g(x, y, \mu) \), where \( f \) and \( g \) be three-times differentiable functions of \( x, y \in \mathbb{R} \), and differentiable in \( \mu \in \mathbb{R} \),

- \((x^*(\mu), y^*(\mu))\) is a fixed point with eigenvalues \( \alpha(\mu) \pm i\omega(\mu) \) where \( \alpha, \omega \in \mathbb{R} \) and \( \omega > 0 \)

- If there exists \( \mu^* \) such that
  - \( \alpha(\mu^*) = 0 \)
  - \( \alpha'(\mu^*) \neq 0 \)
  - \( Q \neq 0 \) where \( Q = a(x^*(\mu^*), y^*(\mu^*), \mu^*) \) and

\[
a(x, y, \mu) = \frac{1}{16} (f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}) + \frac{1}{16\omega(\mu)} (f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}),
\]

- then there is a Hopf bifurcation at \( \mu^* \) which is supercritical if \( Q < 0 \) and subcritical if \( Q > 0 \).
Example 2 (again)

- Recall the modified too-many-twos example

\[ \dot{x} = f(x, y) = x(-2 + \alpha + (1 + \beta)x + (2 + \gamma)y) \]
\[ \dot{y} = g(x, y) = y(2 - 2x - y) \]

- There is a fixed point at \((2/3, 2/3)\), eigenvalues \(\pm 2i/\sqrt{3}\), when \(\alpha = \beta = \gamma = 0\)

- Find that \(a(x, y, \alpha, \beta, \gamma) = \frac{2\beta + \gamma + \beta\gamma}{8\omega}\)

- Eigenvalues

\[
\alpha(3 + \beta) + 2(2\beta + \gamma + \beta\gamma) + \sqrt{-48 + O(\alpha, \beta, \gamma)}
\]
\[
2(3 - \beta + 2\gamma)
\]

- If \(\alpha = 0\), then need \(\gamma + \beta\gamma + 2\beta = 0\), but then \(Q = 0\)
Local Bifurcations

- Have seen four principle bifurcations involving fixed points
- *Local bifurcations* change number or type of fixed points
- Local bifurcations occur at non-hyperbolic fixed points
- Consider $\dot{x} = \alpha + \beta x - x^3$
  - Examining large $x$ implies trapping region
  - Find fixed points
  - Find stability
  - Examine non-hyperbolic fixed points for bifurcations