1. For each of the following pairs of integers \( a, b \) use the Euclidean Algorithm to find \( d = (a, b) \) and find a pair of integers \( x, y \) such that \( ax + by = d \).

(i) \( a = 85, \ b = 41 \);

(ii) \( a = 2613, \ b = 637 \).

**Solution.**

(i)

\[
\begin{array}{c|cc}
85 & 41 \\
2 & 82 & 39 \\
1 & 3 & 2 \\
\end{array}
\]

Thus

\[
1 = 3 - 1.2 \\
= 3 - 1.(41 - 13.3) \\
= 14.3 - 1.41 \\
= 14.(85 - 2.41) - 1.41 \\
= 14.85 - 29.41
\]

So a pair of integers \( x, y \) such that \( 85x + 41y = 1 \) is given by:

\[
x = 14, \ y = -29.
\]

(ii)

\[
\begin{array}{c|cc}
2613 & 637 \\
4 & 2548 & 585 \\
1 & 65 & 52 \\
\end{array}
\]

Thus

\[
13 = 65 - 1.52 \\
= 65 - 1.(637 - 9.65) \\
= 10.65 - 1.637 \\
= 10.(2613 - 4.637) - 1.637 \\
= 10.2613 - 41.637
\]

So a pair of integers \( x, y \) such that \( 2613x + 637y = 13 \) is given by:

\[
x = 10, \ y = -41.
\]
2. Show that if there exist integers \( x, y \) such that \( ax + by = 1 \) then \( (a, b) = 1 \).

**Solution.** Let \( d = (a, b) \). Then, by Property 1, \( d \mid ax + by \). But \( ax + by = 1 \). So \( d \mid 1 \), i.e. \( d \leq 1 \). The gcd is necessarily \( \geq 1 \). Hence \( d = 1 \).

3. Show that \( (3k + 2, 5k + 3) = 1 \) for any integer \( k \).

**Solution.**

\[
(3k + 2, 5k + 3) = (3k + 2, 2k + 1) \quad \text{since} \quad 2k + 1 = 5k + 3 - (3k + 2) \\
= (k + 1, 2k + 1) \quad \text{since} \quad k + 1 = 3k + 2 - (2k + 1) \\
= (k + 1, k) \quad \text{since} \quad k = 2k + 1 - (k + 1) \\
= (1, k) \quad \text{since} \quad 1 = k + 1 - k \\
= 1
\]

Observe that this is just the Euclidean Algorithm applied to algebraic expressions rather than integers.

4. Show that \( (a, a + 2) = 2 \) if \( a \) is even and \( (a, a + 2) = 1 \) otherwise.

**Solution.** Since \( 2 = a + 2 - a \), we have

\[
(a, a + 2) = (a, 2) = \begin{cases} 2 & \text{if} \ 2 \mid a \\ 1 & \text{otherwise} \end{cases}
\]

5. Show that if \( (a, b) = 1 \) then \( (a + b, a - b) = 1 \) or \( 2 \).

**Solution.** Let \( d = (a + b, a - b) \). Then \( d \mid 2a = (a + b) + (a - b) \) and \( d \mid 2b = (a + b) - (a - b) \). Hence

\[
d \mid (2a, 2b) = 2(a, b) = 2.
\]

That is, \( d = (a + b, a - b) \) is either 1 or 2.

6. Find all solutions to the following *Diophantine Equations*.

   (i) \( 2x + 5y = 11 \).  
   (ii) \( 12x + 18y = 50 \).  
   (iii) \( 202x + 74y = 7638 \).

Does equation (iii) have a solution in *positive* integers \( x, y \)?

**Solution.**

(i) Here \((2, 5) = 1\), and since \(2.3 + 5.1 = 11\) we have the general solution

\[
x = 3 + 5t \\
y = 1 - 2t
\]

(This one can be done without using the *Euclidean Algorithm* since the solution is almost obvious.)

(ii) Here \((12, 18) = 6\). So \(6 \mid 12x + 18y\). But \(6 \nmid 50\). Hence there is no solution.
(iii) The problem is equivalent to that of finding solutions of $101x + 37y = 3819$ (eliminating the common factor: 2). Applying the Euclidean Algorithm to 101 and 37 we have:

\[
\begin{array}{ccc}
101 & 37 \\
2 & 74 & 27 & 1 \\
2 & 27 & 10 & 1 \\
2 & 7 & 3 & 1 \\
2 & 6 & & \\
1 & & & \\
\end{array}
\]

Thus

\[
1 = 7 - 2.3 \\
= 7 - 2.(10 - 1.7) \\
= 3.7 - 2.10 \\
= 3.(27 - 2.10) - 2.10 \\
= 3.27 - 8.10 \\
= 3.27 - 8.(37 - 1.27) \\
= 11.27 - 8.37 \\
= 11.(101 - 2.37) - 8.37 \\
= 11.101 - 30.37
\]

So, by the theorem we have

\[
x = 11.3819 + 37t \\
y = -30.3819 - 101t
\]

as the general solution.

(Alternatively) We could instead make the following observations:

\[
\begin{array}{c}
3700 = 100.37 \\
101 = 1.101 \\
18 = (18.11)101 + (18.-30).37 \\
\end{array}
\]

So \ldots

\[
3819 = 199.101 + -440.37
\]

giving the following general solution

\[
x = 199 + 37s \\
y = -440 - 101s
\]

where $s \in \mathbb{Z}$. The advantage of the second approach is that it yields a much smaller starter solution, making it that much easier to determine whether $x, y$ can both be positive. We see that for $y$ to be positive $s \leq -5$, and for $x$ to be positive $s \geq -5$. Thus, if $s = -5$, both $x$ and $y$ are positive, and the solution in this case is:

\[
x = 14 \quad \text{and} \quad y = 65,
\]

and this is the unique solution with this property.
7. A grocer orders apples and oranges at a total cost of $839. If apples cost 25c each and oranges cost 18c each, how many of each type of fruit did the grocer order?

**Solution.** Let $x$ be the number of apples and $y$ be the number of oranges. Then

$$25x + 18y = 839.$$ 

Applying a slight modification of the *Euclidean Algorithm* to 25 and 18 we have

$$\begin{array}{c|cc}
25 & 18 & 14 & 2 \\
18 & 7 & 4 & \\
2 & 8 & \\
-1 & \\
\end{array}$$

Observe we allowed a ‘negative remainder’ at the last step. Sometimes doing this shortcuts a bit of work . . . essentially the *Euclidean Algorithm* still works if we do this. So

$$-1 = 7 - 2.4$$
$$= 7 - 2.(18 - 2.7)$$
$$= 5.7 - 2.18$$
$$= 5.(25 - 1.18) - 2.18$$
$$= 5.25 - 7.18$$

Hence . . . $1 = -5.25 + 7.18$

Observe that

$$\begin{align*}
800 &= 32.25 \\
36 &= 2.18 \\
3 &= (-5.3).25 + (7.3).18 \\
\text{So . . . } 839 &= 17.25 + 23.18
\end{align*}$$

giving the following general solution

$$x = 17 + 18t$$
$$y = 23 - 25t$$

where $t \in \mathbb{Z}$. Since $x, y$ are both positive, we must have $t = 0$. So there are 17 apples and 23 oranges.

8. An apartment block has units at two rates: most rent at $87/week, but a few rent at $123/week. When all are rented the gross income is $8733/week. How many units of each type are there?

**Solution.** Let $x, y$ be the numbers of apartments at the rates $87/week and $123/week, respectively. Then

$$87x + 123y = 8733.$$ 

Cancelling the common factor 3 we have

$$29x + 41y = 2911.$$
Applying the *Euclidean Algorithm* to 29 and 41 we have

\[
\begin{array}{c|ccc}
  & 29 & 41 \\
\hline
 2 & 24 & 29 & 1 \\
 5 & 12 & 2 \\
 2 & 10 & 2 & 1 \\
\end{array}
\]

So

\[
1 = 5 - 2.2 \\
= 5 - 2.(12 - 2.5) \\
= 5.5 - 2.12 \\
= 5.(29 - 2.12) - 2.12 \\
= 5.29 - 12.12 \\
= 5.29 - 12.(41 - 1.29) \\
= 17.29 - 12.41
\]

Observe that

\[
\begin{align*}
2900 &= 100\cdot 29 \\
12 &= -1.29 + 1.41 \\
-1 &= -17.29 + 12.41
\end{align*}
\]

So . . .

\[
2911 = 82.29 + 13.41
\]

giving the following general solution

\[
x = 82 + 41t \\
y = 13 - 29t
\]

where \( t \in \mathbb{Z} \). Since \( x, y \) are both positive, we must have \( t = 0 \) or \( t = -1 \). If \( t = -1 \) then \( x = 41 < 42 = y \), in which case there are more of the $123/week apartments (contrary to the given information). So \( t = 0 \) and there are 82 apartments renting at $87/week and 13 apartments renting at $123/week.

*9. Find all integers \( x, y \) satisfying: \( \frac{1}{x} + \frac{1}{y} = \frac{1}{14} \).

**Solution.** First let us multiply both sides of the given equation by \( xy \). This gives:

\[
y + x = \frac{xy}{14}
\]

Since \( x, y \in \mathbb{Z} \) we must have \( 14 \mid xy \) and hence \( 7 \mid xy \) whence \( 7 \mid x \) or \( 7 \mid y \). Without loss of generality assume \( 7 \mid x \), and write \( x = 7k \). Then

\[
y + 7k = \frac{ky}{2}
\]

and we have \( 2 \mid ky \) and hence \( 2 \mid k \) or \( 2 \mid y \).

*Case 1: \( 2 \mid k \).* Write \( k = 2\ell \), so that

\[
y + 14\ell = \ell y.
\]
Since $\ell \mid 14\ell$ and $\ell \mid \ell y$ we have $\ell \mid y$. Hence $y = \ell m$, say and so

$$m + 14 = \ell m$$

and we have that $m \mid 14$ (i.e. $m \in \{\pm 1, \pm 2, \pm 7, \pm 14\}$) and $\ell = 1 + 14/m$, giving:

$$x = 14(1 + 14/m)$$
$$y = m(1 + 14/m).$$

Furthermore, since $x, y$ are non-zero, $m \neq -14$. Enumerating the possibilities for the pairs $(x, y)$ we get

$$(210, 15), (-182, 13), (112, 16), (-84, 12), (42, 21), (-14, 7), (28, 28).$$

**Case 2:** $2 \parallel k$ and $2 \parallel y$. Write $y = 2s$, so that

$$2s + 7k = ks.$$

Since $k \mid 7k$ and $k \mid ks$, but $2 \parallel k$ we must have $k \mid s$. Write $s = kt$. Then

$$2t + 7 = kt$$

and we have that $t \mid 7$ (i.e. $t \in \{\pm 1, \pm 7\}$) and $k = 2 + 7/t$, giving:

$$x = 7(2 + 7/t)$$
$$y = 2(2 + 7/t)t.$$

Like the previous case, since $x, y$ are non-zero, $t \neq -7$. Enumerating the possibilities for the pairs $(x, y)$ we get

$$(63, 18), (-35, 10), (21, 42).$$

We found the pair $(21, 42)$ (in the reverse order) in Case 1.

Thus the complete list of pairs for $(x, y)$ is

$$(210, 15), (-182, 13), (112, 16), (-84, 12), (42, 21), (-14, 7), (28, 28), (63, 18), (-35, 10),$$

or of course the same pairs in reverse order.

*10. When Jane is one year younger than Betty will be when Jane is half as old as Betty will be when Jane is twice as old as Betty is now, Betty will be three times as old as Jane was when Betty was as old as Jane is now.

One is in her teens and ages are in completed years. How old are they?

**Solution.** Let Jane’s age now be $x$ and Betty’s age now be $y$. Also, we will represent the ages of Jane and Betty at time $i$ by $x_i$ and $y_i$ respectively. Let’s rewrite the given information, including these choices of variables.

When Jane $[\text{is } x_1, \text{ she}]$ is one year younger than Betty will be $[\text{when she is } y_2$ and] when Jane $[\text{is } x_2$ and she] is half as old as Betty will be $[\text{when she is } y_3$ and] when Jane $[\text{is } x_3$ and she] is twice as old as Betty is now $[\text{when she is } y],$ Betty will be $[y_1$ and she will be] three times as old as Jane was when $[\text{she was } x_4$ and when] Betty was $[y_4$ and she was] as old as Jane is now $[\text{when she is } x].$
So we get the following equations:

\[ x_1 = y_2 - 1 \]
\[ x_2 = \frac{1}{2} y_3 \]
\[ x_3 = 2y \]
\[ y_1 = 3x_4 \]
\[ y_4 = x. \]

Now define \( a, b, c, d \) such that \( x_1 = x + a, x_2 = x + b, x_3 = x + c, x_4 = x + d \); so that \( y_1 = y + a, y_2 = y + b, y_3 = y + c, y_4 = y + d \). Hence the above equations become:

\[ x + a = y + b - 1 \]
\[ x + b = \frac{1}{2}(y + c) \]
\[ x + c = 2y \]
\[ y + a = 3(x + d) \]
\[ y + d = x. \]

Now rearrange these equations (and multiply by a factor where appropriate):

\[
\begin{align*}
\frac{1}{2}x + b - \frac{1}{2}c &= y - 1 \\
\frac{1}{2}x + \frac{1}{2}c &= \frac{1}{2}y \\
3x - a + 3d &= y \\
3x - 3d &= 3y.
\end{align*}
\]

Adding these equations we find that \( a, b, c, d \) cancel and we get:

\[ 8\frac{1}{2}x = 6\frac{1}{2}y - 1 \]

i.e.

\[ 13y - 17x = 2. \quad (1) \]

Since \( x, y \) are only allowed to be integers, this equation is an example of a linear Diophantine Equation. A method for solving such an equation is to first apply the Euclidean Algorithm to find the gcd \( d \) of 13 and 17 (which is clearly 1). Tracing the algorithm backwards one can express \( d \) in terms of 13 and 17. Applying the Euclidean Algorithm then (see Notes) we get:

\[
\begin{array}{c|ccc}
3 & 13 & 17 \\
\hline
12 & 13 & 1 \\
1 & 4 & \\
\end{array}
\]

Thus

\[
\begin{align*}
1 &= 13 - 12 \\
&= 13 - 3 \cdot 4 \\
&= 13 - 3 \cdot (17 - 13) \\
&= 4 \cdot 13 - 3 \cdot 17.
\end{align*}
\]
So

\[
2 = 8.13 - 6.17 \\
= 8.13 + 13.17t - 13.17t - 6.17 \\
= 13(8 + 17t) - 17(6 + 13t).
\]

Comparing the above with (1) we see that we have a solution for (1) if

\[
x = 6 + 13t \\
y = 8 + 17t
\]

for some integer \( t \), which by the theorem in the notes is the general solution. We were also given that one girl was in her teens, so that \( t \) must be 1 and

\[
x = 19, \quad y = 25.
\]

Hence Jane is 19 and Betty is 25.

11. Solve the adjacent \textit{alphametic} (an addition in which: each letter stands for a different digit; and left-most digits of a number are not allowed to be 0).

\[
\begin{array}{c}
A \\
A \\
A \\
W \\
H
\end{array}
\begin{array}{c}
H \\
A \\
A
\end{array}
\begin{array}{c}
A \\
A \\
A
\end{array}
\]

\[
\begin{array}{c}
A \\
H \\
A
\end{array}
\begin{array}{c}
H \\
A \\
A
\end{array}
\]

\[
\begin{array}{c}
A \\
H \\
A
\end{array}
\begin{array}{c}
W \\
A \\
G
\end{array}
\]

Solution.

- First let the \textit{carry} from the right column be \( k \) then it is easy to see that

\[
2A + G = 10k \quad (2) \\
H + A + k = 10 \quad (3) \\
A + W + 1 = 10 \quad (4) \\
H = 1 \quad (5)
\]

- Now \( k \) is either 1 or 2 (considering the least and largest values possible for \( A \) and \( G \) ... remembering that \( 2A + G \) is \textit{exactly} a multiple of 10.)

- Since \( H = 1 \) and \( k \in \{1, 2\} \), it follows from (3) that \( A \) is 8 or 7.

- If \( A = 8 \) then by (4) \( W = 1 \), in which case \( H \) and \( W \) are equal (which is not allowed). So \( A \neq 8 \). Hence \( A = 7 \). Therefore \( k = 2 \) and \( G = 6 \), and by (4) \( W = 2 \).

- So finally we get (and check)

\[
\begin{array}{c}
7 \\
7 \\
2 \\
1
\end{array}
\begin{array}{c}
1 \\
7 \\
7 \\
6
\end{array}
\begin{array}{c}
7 \\
7 \\
1 \\
7
\end{array}
\]

- So HAHA = 1717.
12. About all we know of Diophantus’ life is his epitaph from which his age at death is to be deduced:

Diophantus spent one-sixth of his life in childhood, one-twelfth in youth, and another one-seventh in bachelorhood. A son was born five years after his marriage and died four years before his father at half his father’s age.

Solution. Let $x$ be Diophantus’ age at death. Then

$$x - 4 - \left(\frac{1}{6} + \frac{1}{12} + \frac{1}{7}\right)x + 5 = \frac{1}{2}x.$$  

Rearranging, we get

$$\left(\frac{1}{2} - \left(\frac{1}{6} + \frac{1}{12} + \frac{1}{7}\right)\right)x = 9$$

$$\frac{3}{28}x = 9$$

Hence, $x = 9 \cdot \frac{28}{3} = 84$. Diophantus died at the age of 84.

13. Augustus de Morgan, a nineteenth-century mathematician, stated:

I was $x$ years old in the year $x^2$.

When was he born?

Solution. Since $42^2 = 1764$, $43^2 = 1849$ and $44^2 = 1936$, $x = 43$ and $x^2 = 1849$. So Augustus de Morgan was born in 1806.

14. Prove that for every integer $n$:

(i) $3 \mid n^3 - n$;
(ii) $5 \mid n^5 - n$;
(iii) $7 \mid n^7 - n$;
(iv) $11 \mid n^{11} - n$.

Show that $n^9 - n$ is not necessarily divisible by 9. Hint: Try $n = 2$.

What general result is suggested by the above?

Solution.

(i) $3$ divides exactly one of the three consecutive integers $n - 1, n, n + 1$ and

$$n^3 - n = n(n^2 - 1) = n(n - 1)(n + 1).$$

So $3 \mid n^3 - n$.

(ii) $5$ divides exactly one of the five consecutive integers $n - 2, n - 1, n, n + 1, n + 2$. In terms of congruences, exactly one of $n - 2, n - 1, n, n + 1, n + 2$ is congruent to 0 modulo 5. Thus:

$$n^5 - n = n(n^4 - 1) = n(n^2 - 1)(n^2 + 1) = n(n - 1)(n + 1)(n^2 + 1)$$

$$\equiv n(n - 1)(n + 1)(n^2 - 4) \quad (\text{mod } 5)$$

$$\equiv n(n - 1)(n + 1)(n - 2)(n + 2) \quad (\text{mod } 5)$$

$$\equiv 0 \quad (\text{mod } 5)$$

So $5 \mid n^5 - n$. 

9
(iii) Exactly one of \( n - 3, n - 2, n - 1, n, n + 1, n + 2, n + 3 \) is congruent to 0 modulo 7. Thus:

\[
n^7 - n = n(n^6 - 1) = n(n^3 - 1)(n^3 + 1)
\]

\[
= n(n - 1)(n^2 + n + 1)(n + 1)(n^2 - n + 1)
\]

\[
\equiv n(n - 1)(n^2 + n - 6)(n + 1)(n^2 - n - 6) \pmod{7}
\]

\[
\equiv n(n - 1)(n + 3)(n - 2)(n + 1)(n - 3)(n + 2) \pmod{7}
\]

\[
\equiv 0 \pmod{7}
\]

So \( 7 \mid n^7 - n \).

(iv) \( n \) is congruent to exactly one of \(-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\) modulo 11. It is simply a matter of checking, for each congruence possibility of \( n \), that \( n^{11} - n \) (or a factor of \( n^{11} - n \)) is congruent to 0 modulo 11. Note, first that:

\[
n^{11} - n = n(n^{10} - 1).
\]

- If \( n \equiv 0 \pmod{11} \) there is nothing to check since \( n \) is a factor of \( n^{11} - n \).
- \((-1)^{10} - 1 \equiv 1^{10} - 1 \equiv 0 \pmod{11} \).
  So \( n^{10} - 1 \equiv 0 \pmod{11} \) if \( n \equiv \pm 1 \pmod{11} \).
- \( 2^5 = 32 \equiv -1 \pmod{11} \). So \( 2^{10} = (2^5)^2 \equiv 1 \pmod{11} \).
  Hence \((-2)^{10} - 1 \equiv 2^{10} - 1 \equiv 0 \pmod{11} \).
  So \( n^{10} - 1 \equiv 0 \pmod{11} \) if \( n \equiv \pm 2 \pmod{11} \).
- \( 3^5 = 243 \equiv 1 \pmod{11} \). So \( 3^{10} = (3^5)^2 \equiv 1 \pmod{11} \).
  Hence \((-3)^{10} - 1 \equiv 3^{10} - 1 \equiv 0 \pmod{11} \).
  So \( n^{10} - 1 \equiv 0 \pmod{11} \) if \( n \equiv \pm 3 \pmod{11} \).
- \( 2^5 = 32 \equiv -1 \pmod{11} \). So \( 4^{10} = (2^5)^4 \equiv 1 \pmod{11} \).
  Hence \((-4)^{10} - 1 \equiv 4^{10} - 1 \equiv 0 \pmod{11} \).
  So \( n^{10} - 1 \equiv 0 \pmod{11} \) if \( n \equiv \pm 4 \pmod{11} \).
- \( 5^2 = 25 \equiv 4 \pmod{11} \) and \( 4^5 = (2^5)^2 \equiv 1 \pmod{11} \).
  So \( 5^{10} = (5^2)^5 \equiv 4^5 \equiv 1 \pmod{11} \). Hence \((-5)^{10} - 1 \equiv 5^{10} - 1 \equiv 0 \pmod{11} \).
  So \( n^{10} - 1 \equiv 0 \pmod{11} \) if \( n \equiv \pm 5 \pmod{11} \).

So, for each congruence possibility of \( n \), we find a factor of \( n^{11} - n \) is congruent to 0 modulo 11. So for any integer \( n \), \( n^{11} - n \equiv 0 \pmod{11} \). Hence for any integer \( n \mid n^{11} - n \).

Now \( 2^9 - 2 = 510 \) and \( 9 \mid 510 \); so 9 need not divide \( n^9 - n \). The general result suggested by the above is the Corollary to Fermat’s Little Theorem, which may be written in the following way:

If \( n \) is an integer and \( p \) is a prime then \( p \mid n^p - n \).

15. Prove that \( 3^{6n} - 2^{6n} \) is divisible by 35, for every positive integer \( n \).

**Solution.** Let \( N = 3^{6n} - 2^{6n} \). Now \( 35 = \text{lcm}(5, 7) \). So to check that \( 35 \mid N \), it is enough to show that \( 5 \mid N \) and \( 7 \mid N \).

- Firstly,

\[
N = 3^{6n} - 2^{6n} = 9^{3n} - 4^{3n}
\]

\[
\equiv 4^{3n} - 4^{3n} \pmod{5}
\]

\[
\equiv 0 \pmod{5},
\]

and hence \( 5 \mid N \).
• Similarly,

\[ N = 3^{6n} - 2^{6n} = 27^{2n} - 8^{2n} \]
\[ \equiv (-1)^{2n} - 1^{2n} \pmod{7} \]
\[ \equiv 1^n - 1^n \pmod{7} \]
\[ \equiv 0 \pmod{7}, \]

and hence \(7 \mid N\).

Thus, since \(5 \mid N\) and \(7 \mid N\), we have \(35 = \text{lcm}(5, 7)\) divides \(N = 3^{6n} - 2^{6n}\).

*16. What is the final digit of \(7^{77777}\)?

**Solution.** Firstly, we will call an expression of the form

\[ 7^{77777} \]

a *tower* of 7s. Our problem has a tower of 7 7s. Observe that

\[ 7^4 = (7^2)^2 \equiv (-1)^2 \equiv 1 \pmod{10}. \]

Hence, modulo 10,

\[ 7^k \equiv \begin{cases} 
1 & \text{if } k \equiv 0 \pmod{4} \\
7 & \text{if } k \equiv 1 \pmod{4} \\
-1 & \text{if } k \equiv 2 \pmod{4} \\
-7 & \text{if } k \equiv 3 \pmod{4},
\end{cases} \]

where \(k\) is a natural number. Thus to determine the last digit of a tower of 7 7s, we need to determine what a tower of 6 7s is congruent to *modulo* 4. Now, \(7 \equiv -1 \pmod{4}\). Hence, *modulo* 4,

\[ 7^m \equiv \begin{cases} 
1 & \text{if } m \text{ is even} \\
-1 & \text{if } m \text{ is odd},
\end{cases} \]

where \(m\) is a natural number. A tower of 5 7s is certainly odd. So, a tower of 6 7s is congruent to \(-1 \pmod{4}\) (and \(-1 \equiv 3 \pmod{4}\)). So, a tower of 7 7s is congruent to \(-7 \pmod{10}\) (and \(-7 \equiv 3 \pmod{10}\)). Hence, a tower of 7 7s must end in a 3.

17. Prove that for any natural number \(n\) that

\[ 17 \text{ divides } 2^n.3^{2n} - 1. \]

**Solution.**

\[ 2^n.3^{2n} - 1 = (2.3^2)^n - 1 \]
\[ = 18^n - 1 \]
\[ \equiv 1^n - 1 \pmod{17} \]
\[ \equiv 0 \pmod{17}. \]

So \(17 \mid 2^n.3^{2n} - 1\). (Remember: \(N \equiv 0 \pmod{m}\) means exactly the same thing as \(m \mid N\).)
18. Prove that for any natural number $n$

$$17^n - 12^n - 24^n + 19^n$$

is divisible by 35.

**Solution.** Let $N = 17^n - 12^n - 24^n + 19^n$. Now 35 = lcm(5, 7). So to check that 35 $|$ $N$, it is enough to show that 5 $|$ $N$ and 7 $|$ $N$.

• Firstly,

$$N = 17^n - 12^n - 24^n + 19^n$$

$$\equiv 2^n - 2^n - 4^n + 4^n \pmod{5}$$

$$\equiv 0 \pmod{5},$$

and hence 5 $|$ $N$.

• Similarly,

$$N = 17^n - 12^n - 24^n + 19^n$$

$$\equiv 3^n - 5^n - 3^n + 5^n \pmod{7}$$

$$\equiv 0 \pmod{7},$$

and hence 7 $|$ $N$.

Thus, since 5 $|$ $N$ and 7 $|$ $N$, we have 35 = lcm(5, 7) divides $N = 17^n - 12^n - 24^n + 19^n$.

*19. Prove that $5^{99} + 11^{99} + 17^{99}$ is divisible by 33.

**Solution.** Let $N = 5^{99} + 11^{99} + 17^{99}$. Now 33 = lcm(3, 11). So to check that 33 $|$ $N$, it is enough to show that 3 $|$ $N$ and 11 $|$ $N$. Again, we shall use congruences.

• Firstly,

$$N = 5^{99} + 11^{99} + 17^{99}$$

$$\equiv 2^{99} + 2^{99} + 2^{99} \pmod{3}$$

$$\equiv 3 \cdot 2^{99} \pmod{3}$$

$$\equiv 0 \pmod{3},$$

and hence 3 $|$ $N$.

• Similarly,

$$N = 5^{99} + 11^{99} + 17^{99}$$

$$\equiv 5^{99} + 0^{99} + (-5)^{99} \pmod{11}$$

$$\equiv 5^{99} + 0 - 5^{99} \pmod{11}$$

$$\equiv 0 \pmod{11},$$

and hence 11 $|$ $N$.

Thus, since 3 $|$ $N$ and 11 $|$ $N$, we have 33 = lcm(3, 11) divides $N = 5^{99} + 11^{99} + 17^{99}$.
20. What is the final digit of \((((((7^7)^7)^7)^7)^7)^7\)? (7 occurs as a power 10 times.)

**Solution.** The final digit of a (decimal) number is its remainder modulo 10. Now \(7^2 = 49 \equiv -1 \pmod{10}\). So \(7^7 = (7^2)^3 \cdot 7 \equiv -7 \pmod{10}\), and

\[
(7^7)^7 \equiv (-7)^7 \equiv -(7^7) \equiv -(-7) \equiv 7 \pmod{10}.
\]

Proceeding in this way, we see that \(((7^7)^7)^7 \equiv 7 \pmod{10}\), and in general

\[
(\cdots (((7^7)^7)^7)\cdots)^7 \equiv \pm 7 \pmod{10},
\]

where the sign is + if all together there is an even number of 7s appearing as powers in the formula, and − if there is an odd number of 7s appearing as powers in the formula. Now, 10 is even. So the final digit of the given formula is 7.

21. (17th International Olympiad, 1975, Problem 4) When 4444444444444444 is written in decimal notation, the sum of its digits is \(A\). Let \(B\) be the sum of the digits of \(A\). Find the sum of the digits of \(B\).

**Hints:** First show that the sum of the digits of \(B\) is fairly small (in fact: less than 16). Then use the fact that, for any natural number \(N\),

\[
N \equiv \text{(sum of the digits of } N) \pmod{9}.
\]

**Solution.**

- First we will show that the sum of the digits of \(B\) is fairly small. Now 4444 < 10000 = 10^4. Hence

\[
4444444444444444 < 10^4444 = 10^{17776},
\]

and so 44444444 cannot have more than 17776 digits. Thus, \(A\) the sum of the digits of 44444444, cannot be more than 17776.9 = 159,984, (since each digit is at most a 9). Of the natural numbers less than or equal to 159,984, the number with the largest digit sum is 99,999. So \(B\) is not more than 45. Of the natural numbers less than or equal to 45, the number with the largest digit sum is 39. So the sum of the digits of \(B\) is not more than 12.

- Now we use the result given in the hint:

For any natural number \(N\),

\[
N \equiv \text{(sum of the digits of } N) \pmod{9}.
\]

(Note that this result was proved when we proved the divisibility rule for 9.) Using this result we see that 4444444444444444 is congruent to its digit sum \(A\), modulo 9. Using the result again, we see that \(A\) is congruent to its digit sum \(B\), modulo 9. Using the result one further time, we see that \(B\) is congruent to its digit sum, modulo 9. That is,

\[
4444444444444444 \equiv A \pmod{9}
\]

\[
\equiv B \pmod{9}
\]

\[
\equiv \text{(sum of the digits of } B) \pmod{9}.
\]
Now we determine what $4444^{4444}$ is congruent to modulo 9.

$$4444^{4444} \equiv (4 + 4 + 4 + 4)^{4444} \pmod{9}$$
$$\equiv 16^{4444} \pmod{9}$$
$$\equiv (-2)^{4444} \pmod{9}$$
$$\equiv (-2)^{3 \cdot 1481 + 1} \pmod{9}$$
$$\equiv ((-2)^3)^{1481} \cdot (-2) \pmod{9}$$
$$\equiv (-8)^{1481} \cdot (-2) \pmod{9}$$
$$\equiv 1^{1481} \cdot (-2) \pmod{9}$$
$$\equiv 1 \cdot (-2) \pmod{9}$$
$$\equiv 7 \pmod{9}$$

Putting these three facts together we get

$$(\text{the sum of the digits of } B) \equiv 7 \pmod{9}$$

and the sum of the digits of $B$ is a natural number less than or equal to 12. Thus

$$(\text{the sum of the digits of } B) = 7.$$