1. Determine simple rules for divisibility by each of the following natural numbers:

(i) 2
(ii) 3
(iii) 4
(iv) 5
(v) 6
(vi) 8
(vii) 9
(viii) 10
(ix) 11
(x) 12
(xi) 15

Note: there is a rule for 7, but it’s complicated and it is not much better than straight division.

To start you off, here are some rules:

(i) Every natural number $n$ can be written as $10q + r$, where $r$ is the remainder after $n$ is divided by 10, i.e. $r$ is the last digit of $n$. Now $2 \mid 10$. So

$$2 \mid n \text{ if and only if } 2 \mid r,$$

where $r$ is the last digit of $n$. In other words, 2 divides $n$ if and only if $n$ ends in 0 or 2 or 4 or 6 or 8.

(ii) Suppose the decimal representation of $n$ is $a_k a_{k-1} \ldots a_0$. Then

$$n = 10^k a_k + 10^{k-1} a_{k-1} + \cdots + 10a_1 + a_0$$

$$= (10^k - 1)a_k + (10^{k-1} - 1)a_{k-1} + \cdots + 9a_1 + a_k + a_{k-1} + \cdots + a_1 + a_0$$

Now notice that any power of 10 that has 1 subtracted from it, such as $10^k - 1$, is just a string of 9s and so is divisible by 3. Hence,

$$3 \mid n \text{ if and only if } 3 \mid a_k + a_{k-1} + \cdots + a_1 + a_0.$$ 

In other words, 3 divides $n$ if and only if 3 divides the sum of the digits of $n$.

(iii) Every natural number $n$ can be written as $100q + r$, where $r$ is the remainder after $n$ is divided by 100, Now $4 \mid 100$. (Note that 4 does not divide 10.) So

$$4 \mid n \text{ if and only if } 4 \mid r,$$

where $r$ consists of the last two digits of $n$.

(iv) Leave this one for you ;-) 

(v) $6 = \text{lcm}(2, 3)$, so to check divisibility by 6, we check for divisibility by 2 and 3.

2. Show that $x^2 - y^2 = 2$ has no integer solutions.
3. Prove that for every integer $n$:

   (i) $3 \mid n^3 - n$;  
   (ii) $6 \mid n(n-1)(2n-1)$;  
   (iii) $30 \mid n^5 - n$;

4. Prove that for all integers $a$ and $b$: $3$ divides $(a + b)^3 - a^3 - b^3$.

5. (i) Use the Sieve of Eratosthenes to find all primes less than 100.
   
   (ii) What observation can be made that is similar to Observation 2 of the notes?
   
   (iii) There are 10 primes less than 30. How many primes are there that are less than 100?

6. Is 167 prime?

7. Show that if $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ is the prime decomposition of the positive integer $n$, then the number of divisors of $n$ (including 1 and $n$) is $(e_1 + 1)(e_2 + 1) \cdots (e_k + 1)$.

8. Which positive integers have exactly three positive divisors?

9. Which positive integers have exactly four positive divisors?

10. Show that a natural number $n$ is an exact square if and only if it has an odd number of divisors.

*11. There are 50 prisoners in a row of locked cells. With the return of the King from the Crusades, a partial amnesty is declared and it works like this. When the prisoners are still asleep, the jailer walks past the cells 50 times, each time walking from left to right. On the first pass, he turns the lock in every cell (so that every cell is now open). On the second pass he turns the lock on every second cell (meaning that these cells are now locked again). On the third pass, he turns the lock on every third cell, and so on. In general, on the $k$th pass, he turns the lock on every $k$th cell. The question is: which cells are unlocked at the end of the process so that the prisoner is free to go?

12. Is the following statement true or false? *The number $n^2 + n + 41$ is prime for all positive integers $n$.*

13. Is the list of prime numbers finite? i.e. is there a largest prime number?

14. Suppose $p$ is prime.
   
   (i) Show that if $p \mid a^3$ then $p \mid a$.
   
   (ii) Show that if $p \mid b$ and $p \mid a^2 + b^2$ then $p \mid a$.

15. Find the gcd and lcm of 72 and 168.