Algebra – Polynomials: Problems with Solutions

1. The quadratic equation \( x^2 - 3x - 5 = 0 \) has roots \( \alpha, \beta \). Determine \( \alpha^2 + \beta^2 \) and \( \alpha^{-2} + \beta^{-2} \).

Solution. Since

\[(x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta\]

the zeros of \( x^2 - (\alpha + \beta)x + \alpha\beta \) are \( \alpha, \beta \). Comparing coefficients with the quadratic we find

\[
\begin{align*}
\alpha + \beta &= 3 \\
\alpha\beta &= -5.
\end{align*}
\]

(Remember, “\( x^2 - 3x - 5 = 0 \) has roots \( \alpha, \beta \)” means the same thing as “\( x^2 - 3x - 5 \) has zeros \( \alpha, \beta \).”)

Now

\[
\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 3^2 - 2(-5) = 9 + 10 = 19.
\]

\[
\alpha^{-2} + \beta^{-2} = \frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\beta^2 + \alpha^2}{\alpha^2\beta^2} = \frac{\alpha^2 + \beta^2}{\alpha\beta}^2 = \frac{19}{(-5)^2} = \frac{19}{25}.
\]

2. The quadratic polynomial \( x^2 + 4x - 1 \) has zeros \( \alpha, \beta \). Determine \( \alpha^3 + \beta^3 \) and \( \alpha^{-3} + \beta^{-3} \).

Hint. \( (\alpha + \beta)^3 = \alpha^3 + \beta^3 + 3\alpha^2\beta + 3\alpha\beta^2 = \alpha^3 + \beta^3 + 3\alpha\beta(\alpha + \beta) \).

Solution. Since

\[(x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta\]

the zeros of \( x^2 - (\alpha + \beta)x + \alpha\beta \) are \( \alpha, \beta \). Comparing coefficients with the quadratic we find

\[
\begin{align*}
\alpha + \beta &= -4 \\
\alpha\beta &= -1.
\end{align*}
\]
Now
\[
\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)
\]
\[
= (-4)^3 - 3 \cdot -1 \cdot -4
\]
\[
= -64 - 12 = -76
\]
\[
\alpha^{-3} + \beta^{-3} = \frac{1}{\alpha^3} + \frac{1}{\beta^3} = \frac{\beta^3 + \alpha^3}{\alpha^3\beta^3}
\]
\[
= \frac{\alpha^3 + \beta^3}{(\alpha\beta)^3}
\]
\[
= \frac{-76}{(-1)^3} = 76.
\]

3. Solve \(2\left(x + \frac{1}{x}\right)^2 - \left(x + \frac{1}{x}\right) = 10\).

**Solution.** Firstly, let
\[
y = x + \frac{1}{x}
\]
(1)
Then
\[
2\left(x + \frac{1}{x}\right)^2 - \left(x + \frac{1}{x}\right) = 10
\]
(2)
becomes
\[
2y^2 - y = 10.
\]
So
\[
0 = 2y^2 - y - 10
\]
\[
= (2y - 5)(y + 2)
\]
\[
= 2(y - \frac{5}{2})(y + 2).
\]
Hence the solutions for \(y\) are \(y_1 = \frac{5}{2}\) or \(y_2 = -2\).
Rearranging (1) we get
\[
x^2 - xy + 1 = 0
\]
(3)
So, for \(y = y_1\) we get
\[
x^2 - \frac{5}{2}x + 1 = 0.
\]
Hence
\[
0 = 2x^2 - 5x + 2
\]
\[
= (2x - 1)(x - 2)
\]
\[
= 2(x - \frac{1}{2})(x - 2).
\]
So, two solutions of (2) are \(\frac{1}{2}\) or 2.
For \(y = y_2\) in (3) we get
\[
x^2 + 2x + 1 = 0
\]
i.e. \((x + 1)^2 = 0\)
So, \(-1\) is a double root of (2).
Thus, the solutions of (2) are \(-1, \frac{1}{2}\) or 2.
4. Use the Remainder and Factor Theorems to factorise

(i) \( x^3 - 2x^2 - 5x + 6 \)  
(ii) \( x^3 - 5x^2 + 3x + 1 \)

**Solution.** Cubics are in general hard to factorise. So we should try to “pick off” easy factors. In particular, we should look for rational zeros which, if they exist, must be integers since each polynomial is monic.

(i) Let \( p(x) = x^3 - 2x^2 - 5x + 6 \). Looking for integer zeros of \( p(x) \), the candidates are factors of 6, namely: ±1, ±2, ±3, ±6. Let’s try 1 first:

\[
p(1) = 1^3 - 2 \cdot 1^2 - 5 \cdot 1 + 6 = 0.
\]

Since \( p(1) = 0 \), we have \( x - 1 \) is a factor of \( p(x) \), by the Factor Theorem, and after extracting that factor we are left with a quadratic that is easy to factorise:

\[
p(x) = x^3 - 2x^2 - 5x + 6
\]
\[
= (x - 1)(x^2 - x - 6)
\]
\[
= (x - 1)(x - 3)(x + 2)
\]

(ii) Let \( p(x) = x^3 - 5x^2 + 3x + 1 \). Looking for integer zeros of \( p(x) \), this time the only candidates are factors of 1, namely: ±1. Let’s try 1 first:

\[
p(1) = 1^3 - 5 \cdot 1^2 + 3 \cdot 1 + 1 = 0.
\]

Again \( p(1) = 0 \), we have \( x - 1 \) is a factor of \( p(x) \), by the Factor Theorem, and after extracting that factor we are left with a quadratic that we could use the Quadratic Formula to factorise, or the difference of squares technique (which after all is how the Quadratic Formula is derived) demonstrated below:

\[
p(x) = x^3 - 5x^2 + 3x + 1
\]
\[
= (x - 1)(x^2 - 4x + 1)
\]
\[
= (x - 1)(x^2 - 4x + 4 - 3)
\]
\[
= (x - 1)((x - 2)^2 - 3)
\]
\[
= (x - 1)((x - 2) - \sqrt{5})((x - 2) + \sqrt{5})
\]
\[
= (x - 1)(x - 2 - \sqrt{5})(x - 2 + \sqrt{5})
\]

5. The quadratic polynomial \( ax^2 + bx - 4 \) leaves remainder 12 on division by \( x - 1 \) and has \( x + 2 \) as a factor. Find \( a, b \) and the zeros of the polynomial.

**Solution.** Let \( p(x) = ax^2 + bx - 4 \). Then by the Remainder Theorem, since \( p(x) \) leaves remainder 12 on division by \( (x - 1) \),

\[
p(1) = 12
\]
\[
i.e. \ a + b - 4 = 12
\]
\[
a + b = 16 \quad (4)
\]

Similarly, by the Factor Theorem, since \( (x + 2) \) is a factor of \( p(x) \),

\[
p(-2) = 0
\]
\[
i.e. \ 4a - 2b - 4 = 0
\]
\[
2a - b = 2 \quad (5)
\]
Adding (4) and (5) gives
\[3a = 18\]
\[a = 6,\]
and substitution back into (4) gives
\[6 + b = 16\]
\[b = 10.\]

We have already observed that one zero of \(p(x)\) is \(-2\). The product of the zeros of \(p(x)\) is the constant coefficient divided by the leading coefficient, \(-4/a\). Hence the other zero is \(2/a = \frac{1}{3}\). Thus, \(a = 6, b = 10\) and the zeros are: \(-2, \frac{1}{3}\). One should now check that:
\[6x^2 + 10x - 4 = 6(x + 2)(x - \frac{1}{3}).\]

6. Find a quadratic equation with roots \(2 + \sqrt{3}\) and \(2 - \sqrt{3}\).

**Solution.** Let \(\alpha = 2 + \sqrt{3}\) and \(\beta = 2 - \sqrt{3}\). Then
\[\alpha + \beta = 4\]
\[\alpha\beta = (2 + \sqrt{3})(2 - \sqrt{3})\]
\[= 2^2 - (\sqrt{3})^2 = 4 - 3 = 1.\]

So, by Viète’s Theorem,
\[p(x) := x^2 - (\alpha + \beta)x + \alpha\beta\]
\[= x^2 - 4x + 1\]
is a quadratic polynomial with zeros \(2 + \sqrt{3}\) and \(2 - \sqrt{3}\), i.e.
\[x^2 - 4x + 1 = 0\]
is a quadratic equation with roots \(2 + \sqrt{3}\) and \(2 - \sqrt{3}\). (Any nonzero multiple of this equation is also a solution.)

7. June solved a quadratic equation of the form:
\[ax^2 + bx + c = 0\]
and got 2 as a root. Kay switched the \(b\) and the \(c\) and got 3 as a root. What was June’s equation?

**Solution.** Observe that the problem is unchanged if the equation is multiplied by a nonzero constant. So the problem does not have a unique solution (if it has at least one), and we may as well assume \(a = 1\). Let \(p(x) := x^2 + bx + c\). Then June’s equation is \(p(x) = 0\). It has 2 as a root. So, by the Factor Theorem,
\[p(2) = 0\]
i.e. \(2^2 + 2b + c = 0\)
\[4 + 2b + c = 0\]
Now let \( q(x) := x^2 + cx + b \). Then Kay’s equation is \( q(x) = 0 \). It has 3 as a root. So again, by the Factor Theorem,

\[
q(3) = 0
\]

i.e. \( 3^2 + 3c + b = 0 \)

\( 9 + 3c + b = 0 \) \( \quad (7) \)

Multiplying (7) by 2 and subtracting (6) eliminates \( b \) giving

\[
14 + 5c = 0
\]

\[ c = -\frac{14}{5} \]

Substitution back in (7) gives

\[
9 - \frac{3 \cdot 14}{5} + b = 0
\]

\[ b = -\frac{3}{5}. \]

Thus June’s equation could be \( x^2 - \frac{2}{5} x - \frac{14}{5} = 0 \). (5x^2 - 3x - 14 = 0 is just as acceptable as a solution.)

8. The equation \( x^2 + ax + (b + 2) = 0 \) has real roots. What is the least value that \( a^2 + b^2 \) could be?

**Solution.** Since \( x^2 + ax + (b + 2) = 0 \) has real roots, its discriminant is non-negative, i.e.

\[ a^2 - 4(b + 2) \geq 0. \]

Hence

\[ a^2 + b^2 \geq b^2 + 4(b + 2) \]

\[ \geq (b + 2)^2 + 4 \]

\[ \geq 4, \] (since the square \((b + 2)^2\) is non-negative). Thus the least value that \( a^2 + b^2 \) can be is 4. (It is achieved for \( a = 0 \) and \( b = -2 \), in which case the equation has real roots that are both 0.)

9. If \( a, b \) are odd integers, prove that the equation

\[ x^2 + 2ax + 2b = 0 \]

has no rational roots.

**Solution.** Suppose the equation has rational roots then its discriminant \( \Delta := (2a)^2 - 4.2b \) must be a square integer. So

\[ \Delta = 2^2(a^2 - 2b). \]

and hence \( a^2 - 2b \) must be a square integer \( N^2 \), say. So

\[ a^2 - 2b = N^2 \]

\[ a^2 - N^2 = 2b \]

\[ (a - N)(a + N) = 2b \] \( \quad (8) \)
Now the RHS of (8) is divisible by 2, but not 4. On the other hand

\[ a - N \equiv a + N \pmod{2} \]

So either \( a - N \) and \( a + N \) are both odd or they are both even; i.e. the LHS of (8) is either odd or divisible by 4. Thus we have a contradiction. Hence the assumption that the equation had rational roots is false. So the equation has no rational roots. (Note, this argument did not use the fact that \( a \) was odd... so \( a \) need only have been an integer. Also note that if one root is rational then both roots are rational.)

An alternative strategy for proving the result would be to show that \( a^2 - 2b \equiv 3 \pmod{4} \) and that no square integer can be congruent to 3 modulo 4.