1 Introduction

At school when you met the topic of Inequalities you were interested in finding the set of solutions for which a given inequality is satisfied, e.g. you might be asked:

For which \( x \in \mathbb{R} \), is \( x^2 \geq 3x - 2 \).

One technique for solving such an inequality is rearrange it to have right hand side 0, and factorise the resulting left hand quadratic. It’s then straightforward to determine the sign of that left hand side expression, and hence find the solution of the inequality as interval(s) of \( \mathbb{R} \):

\[
\begin{align*}
  x^2 &\geq 3x - 2 \\
  x^2 - 3x + 2 &\geq 0 \\
  (x - 2)(x - 1) &\geq 0 
\end{align*}
\]

Now, we observe that \( (x - 2)(x - 1) \) is positive if both factors are negative or both factors are positive, and is zero when either factor is zero, i.e. the solution is \( x \leq 1 \) or \( x \geq 2 \).

Of course, one needs to start this way to gain some familiarity with how inequations differ from equations.

However, our interest in these lectures is to prove certain Inequalities hold for all \( x \in \mathbb{R} \). One technique for this might be to solve an inequality as above, and show that the solution interval is all of \( \mathbb{R} \), but for the sorts of inequalities with which we will consider, often involving several variables, this is generally not a useful approach. Instead, we will build up an armoury of Standard Inequalities and use these to prove the results we are after. Before we do that, let’s start near the beginning.

2 Symbols and Elementary Rules

No doubt, you are very familiar with the symbols

\[ > \geq < \leq \]

but you probably have not thought much about the rules they obey. Let us start with some properties of real numbers.

- A real number can only be one of positive, negative or 0. Put another way, for a real number \( r \), one of \( r \) or \( -r \) is positive or else \( r = 0 \).

- The sum or product of two positive numbers is positive.

- Of course, for any real number \( r \), \( r + 0 = r \) and \( r \cdot 0 = 0 \).

Now, recognise that \( a > b \) means that \( a - b \) is positive. Also \( a \geq b \) means that either \( a > b \) or \( a = b \). (Sometimes, it is useful to interpret \( a = b \) as: \( a - b \) is 0.) Of course, \( a < b \) means \( b > a \); and \( a \leq b \) means \( b \geq a \).
So now let’s look at some rules that involve \( > \) and \( \geq \) (and \( < \) and \( \leq \)). In each rule \( a, b, c, d \)
are real numbers. The proofs will seem obvious – notice in each case we have used just real number properties (the main ones we use are mentioned above.)

- If \( a > b \) then \( a + c > b + c \). *Note that \( c \) is allowed to be negative.*
  
  **Proof.** Let \( a > b \), i.e. \( a - b \) is positive. Now \( a - b = (a + c) - (b + c) \). So \( (a + c) - (b + c) \) is positive, i.e. \( a + c > b + c \).

- If \( a > b \) and \( c \) is positive then \( ac > bc \).
  
  **Proof.** Let \( a > b \), i.e. \( a - b \) is positive. Also, let \( c \) be positive. Thus, \( (a - b)c = ac - bc \) is positive, i.e. \( ac > bc \).

- If \( a > b \) and \( c \) is negative then \( ac < bc \).
  
  **Proof.** Let \( a > b \) and \( c \) be negative, i.e. \( a - b \) and \( -c \) are positive. Thus, \( (a - b)(-c) = bc - ac \) is positive, i.e. \( bc > ac \) (or equivalently \( ac < bc \)).

- Always \( a^2 \geq 0 \). (The minimum value property of a square.)
  
  **Proof.** If \( a \) is positive then \( a \cdot a = a^2 \) is positive. If \( -a \) is positive then \( (-a) \cdot (-a) = a^2 \) is positive. If \( a = 0 \) then \( a \cdot a = a^2 \) is 0. Hence \( a^2 \) is positive or 0, i.e. \( a^2 \geq 0 \).

- If \( a > b \) and \( b > c \) then \( a > c \). *(Transitivity property)*
  
  **Proof.** Let \( a > b \) and \( b > c \), i.e. \( a - b \) and \( b - c \) are positive. Hence \( (a - b) + (b - c) \) is positive. But \( (a - b) + (b - c) = a - c \). Hence \( a - c \) is positive, i.e. \( a > c \).

- If \( a > b \) and \( c > d \) then \( a + c > b + d \).
  
  **Proof.** Let \( a > b \) and \( c > d \), i.e. \( a - b \) and \( c - d \) are positive. Hence \( (a - b) + (c - d) \) is positive. But \( (a - b) + (c - d) = (a + c) - (b + d) \). Hence \( (a + c) - (b + d) \) is positive, i.e. \( a + c > b + d \).

- If \( 0 < a < b \) then \( \frac{1}{b} > \frac{1}{a} > 0 \).
  
  **Proof.** Exercise.

- If \( 0 < a < 1 \) and \( n \) is a natural number then \( 0 < a^n < 1 \).
  
  **Proof.** Exercise. *(Hint: use Mathematical Induction.)*

Observe that if we let \( a = x/y, \) \( b = 1 \) and \( c = y \) then the second rule becomes:

If \( \frac{x}{y} > 1 \) and \( y \) is positive then \( x > y \).

Thus, we may prove that \( x > y \) by showing either

- \( x - y \) is positive; or

- \( \frac{x}{y} > 1 \) provided that \( y \) is positive.
Example 1.  

(i) If \(x, y\) are distinct positive numbers then  
\[ x^3 + y^3 > x^2y + xy^2. \]

**Proof.** We will show that \((x^3 + y^3) - (x^2y + xy^2)\) is positive. Now  
\[
(x^3 + y^3) - (x^2y + xy^2) = x^3 - x^2y + y^3 - xy^2 = x^2(x - y) + y^2(y - x)
\]
\[
= (x^2 - y^2)(x - y)
\]
\[
= (x + y)(x - y)^2.
\]

Now, by our properties of real numbers and our rules, both \(x + y\) and \((x - y)^2\) are positive, and hence their product is positive, i.e. \(x^3 + y^3 > x^2y + xy^2. \)

(ii) If \(x > y > 0\) then  
\[ 4x^3(x - y) > x^4 - y^4. \]

**Proof.** Since \(x > y > 0\) we have \(x > 0\) (using the transitivity property). Now \(x^4 - y^4 = (x - y)(x + y)(x^2 + y^2)\) and each of \(x - y, x + y\) and \(x^2 + y^2\) is positive. (Check the details!) Hence \(x^4 - y^4\) is positive. We are now in a position to prove the result by showing that  
\[
\frac{4x^3(x - y)}{x^4 - y^4} > 1.
\]

But,  
\[
\frac{4x^3(x - y)}{x^4 - y^4} = \frac{4x^3(x - y)}{(x - y)(x^3 + 2x^2y + xy^2 + y^3)}
\]
\[
= \frac{4x^3}{x^3 + 2x^2y + xy^2 + y^3} \quad \text{since } x - y \neq 0
\]
\[
= \frac{4}{1 + \frac{y}{x} + \frac{y^2}{x^2} + \frac{y^3}{x^3}} \quad \text{since } x \neq 0
\]
\[
> 1
\]

The last step is valid since \(0 < \frac{y}{x} < 1\). (Check all the skipped details!) Thus, we may deduce that \(4x^3(x - y) > x^4 - y^4. \)

\[\square\]

3 **Absolute values**

Absolute values are often most easily treated from a geometric point of view. In particular, *the absolute value of a number measures its distance from 0.* We can extend this idea to interpret  
\[|x - a|\]
as the distance of \(x\) from \(a\). Thus to solve  
\[|x + 1| < 3\]
we may first rewrite it as  
\[|x - (-1)| < 3\]
and interpret it as: *the distance of \(x\) from \(-1\) is less than 3* giving us \(-4 < x < 2. \) (To see this, draw a number line.)

Algebraically, we have the following definition for \(|x|\),

\[|x| = \begin{cases} 
  x, & \text{if } x \geq 0, \\
  -x, & \text{if } x < 0. 
\end{cases}\]

and note that for a positive real number \(a\) we have that  
\[|x| < a \quad \text{if and only if } -a < x < a.\]

To gain some familiarity with manipulating absolute values, try the following exercises.
4 Triangle Inequality

Exercises.

1. Find the solution interval(s) for the following inequalities.
   
   (i) \(|x + 7| > 3\)
   
   (ii) \(|2x - 7| < 2\)
   
   (iii) \(|x - 2| \geq |2x + 3|\)
   
   (iv) \(1 - x \geq |x - 1|\)

You will find these exercises solved among some others in my Algebra – Inequalities: Problems with Some Solutions at: http://school.maths.uwa.edu.au/~gregg/Academy/2007/

4 Triangle Inequality

The name of this inequality comes from the geometric observation that the length of a side of triangle must lie between the difference and sum of the other two sides:

Theorem (Triangle Inequality).

\[
||x| - |y|| < |x + y| < |x| + |y|
\]

for any real numbers \(x, y\).

5 Squares are never negative

We identified this property earlier, but it’s so important it bears repeating and putting it in its own section.

The square of a real number is never negative, i.e.

\[x^2 \geq 0, \text{ with equality } \iff x = 0,\]

or more generally

\[x_1^2 + x_2^2 + \cdots + x_n^2 \geq 0, \text{ with equality } \iff x_1 = x_2 = \cdots = x_n = 0.\]

Exercises – squares are non-negative.

2. Prove that for any non-negative \(a, b\),

\[
\frac{a + b}{2} \geq \sqrt{ab}.
\]

This result is AM-GM (which we will discuss further later) for the case \(n = 2\).

3. For arbitrary \(a, b, c \in \mathbb{R}\), prove \(a^2 + b^2 + c^2 \geq ab + bc + ca\).

4. (1990 USSR MO) Prove that for arbitrary \(t \in \mathbb{R}\), the inequality \(t^4 - t + \frac{1}{2} > 0\) holds.

5. Let \(a, b, c, d \in \mathbb{R}\). Prove that the numbers \(a - b^2, b - c^2, c - d^2, d - a^2\) cannot all be larger than \(\frac{1}{4}\).

6. Prove that \((a + 5b)(3a + 2b) \geq (a + 9b)(2a + b)\) for all \(a, b \in \mathbb{R}\).

7. Prove \((p + 2)(q + 2)(p + q) \geq 16pq\) for all \(p, q > 0\).

8. Prove that \(a^2(1 + b^4) + b^2(1 + a^4) \leq (1 + a^4)(1 + b^4)\) for all \(a, b \in \mathbb{R}\).

9. If \(x, y, z \in \mathbb{R}\) such that \(x + y + z = 1\), prove that \(x^2 + y^2 + z^2 \geq \frac{1}{3}\).