Local symmetry properties of graphs

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Automorphisms of graphs

Γ a finite simple connected graph.
Unless otherwise stated, each vertex has valency at least 3.
Vertex set $V\Gamma$, edge set $E\Gamma$.

An automorphism of Γ is a permutation of the vertices which maps edges to edges.

$\text{Aut}(\Gamma)$ is the group of all automorphisms of Γ.
Automorphisms of the Petersen graph

Rotations and reflections gives $D_{10}$.
Interchange inside with outside.
This gives 20 automorphisms.

$\text{Aut}(\Gamma) = S_5$
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$\text{Aut}(\Gamma) = S_5$
Vertex-transitive graphs

Say $\Gamma$ is **vertex-transitive** if $\text{Aut}(\Gamma)$ acts transitively on $V\Gamma$, that is, for any two vertices $v$ and $w$ there is an automorphism $g$ mapping $v$ to $w$.

The Petersen graph is vertex-transitive.

Such graphs are regular, for $g$ induces a bijection from $\Gamma(v)$ to $\Gamma(w)$. 
The Frucht graph is regular but has trivial automorphism group.
Edge-transitive graphs

Say $\Gamma$ is edge-transitive if $\text{Aut}(\Gamma)$ acts transitively on $E\Gamma$.

The Petersen graph is edge-transitive.

Suppose that $\Gamma$ is edge-transitive but not vertex-transitive.

Then each vertex-orbit contains a unique vertex from each edge.

Thus only two orbits of vertices and these are the two biparts.
Folkman graph

Edge-transitive, regular but not vertex-transitive.
We say \( \Gamma \) is **arc-transitive** if \( \text{Aut}(\Gamma) \) acts transitively on the set \( A\Gamma \) of arcs, that is on all ordered pairs of adjacent vertices.

The Petersen graph is arc-transitive.

Arc-transitive implies edge-transitive and vertex-transitive.

Vertex- and edge-transitive but not arc-transitive graphs are called **half-arc-transitive**.

Holt graph
Interaction

vertex-transitive

half-arc-transitive

Arc-transitive

edge-transitive
Coset graphs

• $G$ a group with subgroup $H$,

• $g \in G \backslash H$ such that $g^2 \in H$.

We can construct the graph $Cos(G, H, HgH)$ with

- vertex set: cosets of $H$ in $G$
- adjacency: $Hx \sim Hy$ if and only if $xy^{-1} \in HgH$

$G$ acts by right multiplication on vertices and is transitive on $A\Gamma$.

Any arc-transitive graph $\Gamma$ can be constructed in this way:

• $G = \text{Aut}(\Gamma), H = G_v$

• $g$ an element interchanging $v$ and $w$, where $\{v, w\} \in E\Gamma$.

Petersen graph: $G = S_5$, $H = G_{\{1,2\}}$ and $g = (13)(24)$. 
Coset graphs II

- a group $G$ with subgroups $L$ and $R$

We can construct the bipartite graph $\text{Cos}(G, L, R)$ with

vertex set: cosets of $L$ in $G$ and cosets of $R$ in $G$

adjacency: $Lx \sim Ry$ if and only if $Lx \cap Ry \neq \emptyset$

or equivalently, if $xy^{-1} \in LR$

$G$ acts by right multiplication with two orbits on vertices and
transitive on $E\Gamma$.

Any edge-transitive bipartite graph can be constructed in this way:
$G = \text{Aut}(\Gamma)$, $L = G_v$ and $R = G_w$ for some edge $\{v, w\}$. 
s-arc transitive graphs

An s-arc in a graph is an \((s + 1)\)-tuple \((v_0, v_1, \ldots, v_s)\) of vertices such that \(v_i \sim v_{i+1}\) and \(v_{i-1} \neq v_{i+1}\).
s-arc transitive graphs

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\[\text{Diagram of a graph with 4 vertices and 5 edges.}\]
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\[
\begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{graph.png}}
\end{array}
\]
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A graph is s-arc transitive if \(\text{Aut}(\Gamma)\) is transitive on the set of s-arcs.
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A graph is s-arc transitive if \(\text{Aut}(\Gamma)\) is transitive on the set of s-arcs.

\(K_4\) is 2-arc transitive but not 3-arc transitive.
Some basic facts

$s$-arc transitive implies $(s - 1)$-arc transitive.

In particular, $s$-arc transitive implies arc-transitive and hence vertex-transitive.

If $G \leqslant \text{Aut}(\Gamma)$ such that $G$ acts transitively on $s$-arcs we say that $\Gamma$ is $(G, s)$-arc transitive.
Examples

- Cycles are $s$-arc transitive for arbitrary $s$.
- Complete graphs are 2-arc transitive.
- Petersen graph is 3-arc transitive.
- Heawood graph (point-line incidence graph of Fano plane) is 4-arc transitive.
- Tutte-Coxeter graph (point-line incidence graph of the generalised quadrangle $W(3, 2)$) is 5-arc transitive.
Bounds on $s$

**Tutte (1947, 1959):** For cubic graphs, $s \leq 5$.

**Weiss (1981):** For valency at least 3, $s \leq 7$.

Upper bound is met by the generalised hexagons associated with $G_2(q)$ for $q = 3^n$.

These are bipartite, with valency $q + 1$ and $2(q^5 + q^4 + q^3 + q^2 + q + 1)$ vertices.
Local action

Γ is $(G, 2)$-arc transitive if and only if $G_v$ is 2-transitive on $\Gamma(v)$ and $G$ transitive on $V\Gamma$. 
Local action

\( \Gamma \) is \((G, 2)\)-arc transitive if and only if \( G_v \) is 2-transitive on \( \Gamma(\nu) \) and \( G \) transitive on \( V\Gamma \).
Γ is \((G, 2)\)-arc transitive if and only if \(G_v\) is 2-transitive on \(\Gamma(v)\) and \(G\) transitive on \(V\Gamma\).
Local action

Γ is \((G, 2)\)-arc transitive if and only if \(G_v\) is 2-transitive on \(\Gamma(\nu)\) and \(G\) transitive on \(V\Gamma\).
Structure of vertex stabiliser

**Tutte:** For a cubic graph which is $s$-arc transitive but not $(s + 1)$-arc transitive, $|G_v| = 3.2^{s-1}$.

**Djoković and Miller (1980):** Determined the possible structures of a vertex stabiliser in cubic case: only 7 possibilities.

Use knowledge of 2-transitive groups to study possible vertex stabilisers.
$\mathcal{B}$ a partition of $V\Gamma$

Quotient graph $\Gamma_\mathcal{B}$:
- vertex set: parts of $\mathcal{B}$
- adjacency: $B_1 \sim B_2$ if there exists $v_1 \in B_1$ and $v_2 \in B_2$ such that $v_1 \sim v_2$.

$\Gamma$ is a cover of $\Gamma_\mathcal{B}$ if:
The quotient of a 2-arc transitive graph is not necessarily 2-arc transitive.

Babai (1985): Every finite regular graph has a 2-arc transitive cover.
Instead look at normal quotients, that is, where $B$ is the set of orbits of some normal subgroup $N$ of $G \leq \text{Aut}(\Gamma)$.

Denote by $\Gamma_N$.

**Theorem (Praeger 1993)**

Let $\Gamma$ be a $(G, s)$-arc transitive graph and $N \triangleleft G$ with at least three orbits on $V\Gamma$. Then $\Gamma_N$ is $(G, s)$-arc transitive. Moreover, $\Gamma$ is a cover of $\Gamma_N$.

So the basic $(G, s)$-arc transitive graphs to study are those for which all nontrivial normal subgroups of $G$ have at most two orbits.
Quasiprimitive groups

A permutation group is quasiprimitive if every nontrivial normal subgroup is transitive.

Praeger (1993) proved an O’Nan-Scott Theorem for quasiprimitive groups which classifies them into 8 types.

Only 4 are possible for a 2-arc transitive group of automorphisms.

- Twisted Wreath: Baddeley (1993)
- Product Action: Li-Seress (2006+)
- Almost Simple:

Li (2001): 3-arc transitive implies Almost Simple or Product Action.
Bipartite case

Let $\Gamma$ be a bipartite graph with group $G$ acting transitively on $V\Gamma$. $G$ has an index 2 subgroup $G^+$ which fixes the two halves setwise. In particular, $G$ cannot be quasiprimitive.

The basic graphs to study are those where every normal subgroup of $G$ has at most two orbits, ie $G$ is biquasiprimitive on vertices.


In fact $G^+$ may or may not be quasiprimitive on each orbit.

See Alice Devillers’ talk.
Locally $s$-arc transitive

In the bipartite graph case, the index two subgroup $G^+$ contains each vertex stabiliser $G_v$.

In particular, $(G^+)_v = G_v$ and so $(G^+)_v$ acts transitively on the set of all $s$-arcs starting at $v$.

We say that $\Gamma$ is locally $(G, s)$-arc transitive if for all vertices $v$, $G_v$ acts transitively on the set of $s$-arcs starting at $v$.

- $G_v$ is 2-transitive on $\Gamma(v)$.
- If $G$ is transitive on vertices then $\Gamma$ is $(G, s)$-arc-transitive.
- If $G$ is intransitive on vertices then $G$ has two orbits and $\Gamma$ is bipartite.

eg point-line incidence graph of a projective space
Bounds on $s$

Stellmacher (1996): $s \leq 9$

Bound attained by classical generalised octagons associated with $^{2}F_{4}(q)$ for $q = 2^n$, $n$ odd.

These have valencies $\{2^n + 1, 2^{2n} + 1\}$.

Main approach of study has been to determine possibilities for $G^{\Gamma(v)}_v$ and $G^{\Gamma(w)}_w$ for some edge $\{v, w\}$ and try to determine $\{G, G_v, G_w\}$. 
Global approach

Theorem (Giudici-Li-Praeger (2004))

• $\Gamma$ a locally $(G, s)$-arc transitive graph,
• $G$ has two orbits $\Delta_1, \Delta_2$ on vertices,
• $N \triangleleft G$.

1. If $N$ intransitive on both $\Delta_1$ and $\Delta_2$ then $\Gamma_N$ is locally $(G/N, s)$-arc transitive. Moreover, $\Gamma$ is a cover of $\Gamma_N$.
2. If $N$ transitive on $\Delta_1$ and intransitive on $\Delta_2$ then $\Gamma_N$ is a star.
Basic graphs

There are two types of basic locally \((G, s)\)-arc transitive graphs:

(i) \(G\) acts faithfully and quasiprimitively on both \(\Delta_1\) and \(\Delta_2\).

(ii) \(G\) acts faithfully on both \(\Delta_1\) and \(\Delta_2\) and quasiprimitively on only \(\Delta_1\). (The star case)

Theorem (Giudici-Li-Praeger (2004))

1. In case \((i)\), either
   - the quasiprimitive types of \(G^{\Delta_1}\) and \(G^{\Delta_2}\) are the same and one of 4 possibilities, or
   - one is Simple Diagonal while the other is Product Action.

2. In case \((ii)\) there are only 5 possibilities for the type of \(G^{\Delta_1}\).
The $\{SD, PA\}$ case

All characterised by Giudici-Li-Praeger (2006-07).

Either $s \leq 3$ or the following locally 5-arc transitive example:

$\Gamma = \text{Cos}(G, L, R)$ with

- $G = \text{PSL}(2, 2^m)^{2^m} \rtimes \text{AGL}(1, 2^m)$, $m \geq 2$,
- $L = \{(t, \ldots, t) \mid t \in \text{PSL}(2, 2^m)\} \times \text{AGL}(1, 2^m)$,
- $R = (C_{2^m} \rtimes C_{2^m - 1}) \rtimes \text{AGL}(1, 2^m)$

On the set of cosets of $R$, $G$ preserves a partition into $(2^m + 1)^{2^m}$ parts.

- valencies $\{2^m + 1, 2^m\}$
- $G_v^{\Gamma(v)} = \text{PSL}(2, 2^m)$, $G_w^{\Gamma(w)} = \text{AGL}(1, 2^m)$

Important place in the Stellmacher/van Bon program.
Distance transitive graphs

Γ is called **distance transitive** if for each $i$, $\text{Aut}(\Gamma)$ is transitive on the set $\{(v, w) \mid d(v, w) = i\}$.

A graph satisfying these regularity properties is called **distance regular**.

Shrikhande graph
An imprimitive distance transitive graph is either bipartite or antipodal.

In bipartite case, the distance two graph $\Gamma^{(2)}$ has two connected components, each distance-transitive.

In the antipodal case, the antipodal quotient is distance-transitive.
Primitve distance transitive graphs


- can be derived from a Hamming graph, or
- is of Almost Simple or Affine type.

Classification is almost complete.
Locally distance transitive graphs

Say $\Gamma$ is \textbf{locally distance transitive} if for each vertex $v$ and integer $i$, $\text{Aut}(\Gamma)_v$ acts transitively on the set of vertices at distance $i$ from $v$.

- If $\Gamma$ is vertex-transitive then it is distance transitive.
- If $\Gamma$ is not vertex-transitive then $\text{Aut}(\Gamma)$ has two orbits on vertices and $\Gamma$ is bipartite.

The distance parameters for a vertex only depend on the part of the bipartition it belongs to.

eg line-plane incidence graph of a projective space.
If $\Gamma$ is locally distance transitive and bipartite then $\Gamma^{(2)}$ has two connected components, each of which is distance transitive.

In the nonregular case, at least one is primitive.

So use knowledge of primitive distance transitive graphs.
Locally $s$-distance transitive and $s$-distance transitive

Joint work with Alice Devillers, Cai Heng Li, Cheryl Praeger

$\Gamma$ is called locally $(G, s)$-distance transitive if $s \leq \text{diam}(\Gamma)$, and for each vertex $v$ and $i \leq s$, $G_v$ acts transitively on $\Gamma_i(v)$.

A $(G, s)$-distance transitive graph is a locally $(G, s)$-distance transitive graph such that $G$ is transitive on $V\Gamma$.

If $s \leq \lfloor \frac{g-1}{2} \rfloor$, where $g$ is the length of the shortest cycle, then $\Gamma$ is (locally) $s$-distance transitive if and only if $\Gamma$ is (locally) $s$-arc transitive.
Quotienting?

In bipartite case, the connected components of \( \Gamma^{(2)} \) have half the diameter of \( \Gamma \).

Paths in \( \Gamma \) may decrease in length in \( \Gamma_N \) and indeed \( \Gamma_N \) may have smaller diameter than \( \Gamma \).
Quotienting

Let $LDT(s)$ be the set of graphs $\Gamma$ that are locally $s'$-distance transitive where $s' = \min\{s, \text{diam}(\Gamma)\}$.

Theorem (Devillers-Giudici-Li-Praeger)

Let $s \geq 2$ and let $\Gamma \in LDT(s)$ relative to $G$ and let $N \triangleleft G$ with at least three orbits on vertices. Then one of the following holds:

- $\Gamma = K_{m[b]}$,  
- $\Gamma_N$ is a star,  
- $\Gamma_N \in LDT(s)$ relative to $G/N$ and $\Gamma$ is a cover of $\Gamma_N$.  

Basic graphs

There are four types of basic locally \((G, s)\)-distance transitive graphs to study:

- \(G\) acts quasiprimatively on \(V\Gamma\);
- \(\Gamma\) is bipartite, \(G\) is biquasiprimitive on \(V\Gamma\) and \(G^+\) acts faithfully on each orbit;
- \(\Gamma\) is bipartite, \(G = G^+\) acts faithfully and quasiprimitively on each orbit;
- \(\Gamma\) is bipartite, \(G = G^+\) acts faithfully on both orbits and quasiprimitively on only one.

These are currently under investigation.