Stabilizer theorems for even cut matroids

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Abstract

A graft is a representation of an even cut matroid $M$ if the cycles of $M$ correspond to the even cuts of the graft. Two, long standing, open questions regarding even cut matroids are the problem of finding an excluded minor characterization and the problem of efficiently recognizing this class of matroids. Progress on these problems has been hampered by the fact that even cut matroids can have an arbitrary number of pairwise inequivalent representations (two grafts are equivalent if the underlying graphs are related by Whitney-flips and the grafts have the same $T$-joins). We show that we can bound the number of inequivalent representations of an even cut matroid $M$ (under some connectivity assumptions) if $M$ contains any fixed size minor that is not a projection of a co-graphic matroid. For instance, any connected even cut matroid which contains $R_{10}$ as a minor has at most 10 inequivalent representations.

1 Introduction

We assume that the reader is familiar with the basics of matroid theory. See Oxley [5] for the definitions of the terms used here. We will only consider binary matroids in this paper. Thus the reader should substitute the term “binary matroid” every time “matroid” appears in this text.

In this article, we will consider graphs with multiple edges and loops. Let $G$ be a graph. Given a set of vertices $U$, we denote by $\delta_G(U)$ the cut induced by $U$, that is $\delta_G(U) := \{(u,v) \in E(G) : u \in U, v \not\in U\}$. We denote by cut$(G)$ the set of all cuts of $G$. Since the cuts of $G$ correspond to the cycles
of the co-graphic matroid of $G$, we identify $\text{cut}(G)$ with that matroid and say that $G$ is a representation of that matroid.

A graft is a pair $(G, T)$ where $G$ is a graph, $T \subseteq V(G)$ and $|T|$ is even. The vertices in $T$ are the terminals of the graft. A cut $\delta(U)$ is $T$-even (respectively $T$-odd) if $|T \cap U|$ is even (respectively odd). When there is no ambiguity we omit the prefix $T$ when referring to $T$-even and $T$-odd cuts. We denote by $\text{ecut}(G, T)$ the set of all even cuts of $(G, T)$. It can be verified that $\text{ecut}(G, T)$ is the set of cycles of a binary matroid, which is called the even cut matroid represented by $(G, T)$ [7, 15]. We identify $\text{ecut}(G, T)$ with that matroid and say that $(G, T)$ is a representation of that matroid. Observe that since $\text{cut}(G) = \text{ecut}(G, \emptyset)$, every co-graphic matroid is an even cut matroid.

1.1 Representations of co-graphic matroids are nice

We will state a theorem that shows, for a co-graphic matroid, how to construct the set of all its representations (as graphs) from a single representation. We require a number of definitions.

Let $G$ be a graph. For a set $X \subseteq E(G)$, we write $V_G(X)$ to refer to the set of vertices incident to an edge of $X$ and $G[X]$ for the subgraph with vertex set $V_G(X)$ and edge set $X$. Let $G$ be a graph and let $X \subseteq E(G)$. We write $R_G(X)$ for $V_G(X) \cap V_G(\bar{X})$. \(^1\) Suppose that $R_G(X) = \{u_1, u_2\}$ for some $u_1, u_2 \in V(G)$. Let $G'$ be the graph obtained by identifying vertices $u_1$ and $u_2$ of $G[X]$ with vertices $u_2$ and $u_1$ of $G[\bar{X}]$ respectively. Then $G'$ is obtained from $G$ by a Whitney-flip on $X$. We will also call Whitney-flip the operation consisting of identifying two vertices from distinct components, or the operation consisting of partitioning the graph into components each of which is a block of $G$. We define two graphs to be equivalent if one can be obtained from the other by a sequence of Whitney-flips (it is easy to verify that this does indeed define an equivalence relation).

In a seminal paper [14], Whitney proved the following.

**Theorem 1.** A co-graphic matroid has a unique representation, up to equivalence.

It follows in particular that, if a co-graphic matroid is 3-connected, then it has a unique representation.

1.2 Representations of even cut matroids are naughty

The situation is considerably more complicated for even cut matroids than for co-graphic matroids, as we will illustrate in this section.

\[^1\] $\bar{X} = E(G) - X$, where, for any pair of sets $A$ and $B$, $A - B = \{a \in A : a \notin B\}$. Throughout the paper we shall omit indices when there is no ambiguity. For instance we may write $R(X)$ for $R_G(X)$.  

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Given a graph $H$, we denote by $V_{odd}(H)$ the set of vertices of $H$ of odd degree. Given a graft $(G, T)$ we say that $J \subseteq E(G)$ is a $T$-join of $G$ if $T = V_{odd}(G[J])$. Note that, if $J$ is a $T$-join of $G$, then a cut $C$ of $G$ is $T$-even if and only if $|C \cap J|$ is even. We say that two grafts $(G_1, T_1)$ and $(G_2, T_2)$ are \textit{equivalent} if $G_1$ and $G_2$ are equivalent and a $T_1$-join of $G_1$ is a $T_2$-join of $G_2$. It is easy to see that, given equivalent graphs $G_1$ and $G_2$, for two sets of terminals $T_1$ for $G_1$ and $T_2$ for $G_2$ we have $\text{ecut}(G_1, T_1) = \text{ecut}(G_2, T_2)$ if and only if $(G_1, T_1)$ and $(G_2, T_2)$ are equivalent. Equivalence of grafts does indeed define an equivalence relation. It follows that we can partition the representations of any even cut matroid $N$ into equivalence classes $\mathbb{F}_1, \ldots, \mathbb{F}_k$. We will say that $\mathbb{F}_i (i \in [k])$ is an \textit{equivalence class} of $N$. There is no direct analogue to Whitney’s theorem for even cut matroids, as the following result illustrates.

\textbf{Remark 2.} \textit{For any integer $k$, there exists an even cut matroid $M$ with $|E(M)| \leq 6k$ and $4^{k-1}$ equivalence classes.}

We now describe a general operation to construct a matroid as in the previous result. Let $(G, T)$ be a graft with $T = \{a, b, c, d\}$, for distinct vertices $a, b, c, d$. Let $X \subseteq E(G)$ with $\mathcal{R}_G(X) \subseteq T$. Construct a graph $G'$ by:

- identifying vertex $a$ of $G[X]$ with vertex $b$ of $G[\bar{X}]$ to a vertex $a'$;
- identifying vertex $b$ of $G[X]$ with vertex $a$ of $G[\bar{X}]$ to a vertex $b'$;
- identifying vertex $c$ of $G[X]$ with vertex $d$ of $G[\bar{X}]$ to a vertex $c'$;
- identifying vertex $d$ of $G[X]$ with vertex $c$ of $G[\bar{X}]$ to a vertex $d'$.

Let $T' = \{a', b', c', d'\}$. We say that the grafts $(G, T)$ and $(G', T')$ are related by a \textit{shuffle move} on $X$ with \textit{pairing} $a, b$. A subset $C$ of edges of $G$ is a \textit{cycle} if $G[C]$ is a graph where every vertex has even degree. Note that every cycle and every $T$-join of $G$ corresponds to either a cycle or a $T'$-join of $G'$ and vice versa. As cycles and $T$-joins of $(G, T)$ form the co-cycles of $\text{ecut}(G, T)$ (see [3]) this implies that $\text{ecut}(G, T) = \text{ecut}(G', T')$.

Using shuffle moves we can construct inequivalent grafts representing the same even cut matroid. An example is given in Figure 1: (a) and (b) are non-equivalent representations of the same even cut matroid. White vertices are terminals, dotted lines denote vertices that are identified. Each of the graphs $G_1, \ldots, G_4$ may be any graph. The arrows indicate how each piece is flipped. For all $i \in [4]$ $^2[k] = \{1, \ldots, k\}$
let $X_i$ be the set of edges for which $G_i = G[X_i]$. We may obtain (b) from (a) by applying three shuffle moves, on sets $X_2$, $X_3$ and $X_4$ (each with a different pairing). This operation generalizes to any number of separations $X_1, \ldots, X_k$ and we may choose to apply any number of shuffle moves. Thus, for any given $k$, it is easy to construct an even cut matroid $M$ satisfying Remark 2. For example, we may choose each $X_i$ to be a copy of the graph $K_4$. Then, for every $i \neq 1$, we may apply a shuffle move on $X_i$ or not. For a shuffle move we have three different choices of pairings. This leads to $4^{k-1}$ inequivalent representations of $M$.

As Remark 2 shows, if a graft $(G, T)$ has only four terminals, then $\text{ecut}(G, T)$ may have many inequivalent representations. One may wonder if having more than four terminals forces the representation to be unique, up to equivalence. Unfortunately, this is not the case, as stated in the following remark.

**Remark 3.** For every integer $k$, there exists a graft $(G, T)$ with the property that:

1. every graft equivalent to $(G, T)$ has at least $k$ terminals, and
2. $\text{ecut}(G, T)$ has at least two inequivalent representations.

An example of a construction for Remark 3 for $k = 12$ is given in Figure 2: (a) and (b) are non-equivalent representations of the same even cut matroid. White vertices are terminals, dashed lines represent 2-separations. Each of the graphs $G_1, \ldots, G_5$ may be any graph. The arrows indicate how each piece is flipped. Note that every cycle and every $T$-join of (a) corresponds to either a cycle or a $T$-join of (b) and vice versa. As cycles and $T$-joins form the co-cycles of even cut matroids, this implies
that (a) and (b) represent the same even cut matroid. This operation generalizes to any number of graphs $G_1, \ldots, G_r$ in which case we obtain $2r + 2$ terminals in both (a) and (b). This proves Remark 3. We will see that this construction is a special example of the clip operation defined in section 3.2. It is also possible to extend this construction to graphs that are 4-connected.

1.3 Main results

Given a matroid $M$ and disjoint subsets $I, J \subseteq E(M)$, the matroid $M \setminus I / J$ denotes the minor of $M$ obtained by deleting the elements in $I$ and contracting the elements in $J$. We define minor operations on grafts as follows. Let $(G, T)$ be a graft and let $e \in E(G)$. Then $(G, T) \setminus e$ is defined as $(G \setminus e, T')$, where $T' = \emptyset$ if $e$ is an odd bridge of $(G, T)$ and $T' = T$ otherwise. \textsuperscript{3} We define $(G, T)/e$ as $(G/e, T')$, where $T'$ is defined as follows. Let $u$ and $v$ be the ends of $e$ in $G$ and let $w$ be the vertex obtained by contracting $e$. If $x \neq w$, then $x \in T'$ if and only if $x \in T$; $w \in T'$ if and only if $|\{u, v\} \cap T| = 1$. With this definition we have that

**Remark 4.** $\text{ecut}(G, T)/C \setminus D = \text{ecut}((G, T)/D \setminus C)$.

In particular, this implies that being an even cut matroid is a minor closed property.

\textsuperscript{3}Given a graph $H$ and $e \in E(H)$, $G \setminus e$ is the graph obtained by deleting $e$, whereas $G/e$ is the graph obtained by contracting $e$.}
1.3.1 Non-degenerate minors

We say that a graft \((G, T)\) is degenerate if \(|T'| \leq 4\) for some graft \((G', T')\) equivalent to \((G, T)\). An even cut matroid \(M\) is degenerate if some representation \((G, T)\) of \(M\) is degenerate, it is non-degenerate otherwise. If a graft has less than six terminal, then so do all of its minors. It follows from Remark 4 that being degenerate is a minor closed property. If a matroid is co-graphic, then it has a representation \((G, \emptyset)\) as an even cut matroid, hence it is degenerate.

An example of an even cut matroid which is non-degenerate is given by the matroid \(R_{10}\) (introduced in \([10]\)). \(R_{10}\) has, up to equivalence, 10 representations as an even cut matroid, which are all isomorphic to the graft in Figure 3. (Terminals in that figure are presented by white vertices.) Hence \(R_{10}\) is a non-degenerate even cut matroid.

![Figure 3: Graft representation of \(R_{10}\).](image)

We are now ready to present the first main result of the paper.

**Theorem 5.** Let \(M\) be a 3-connected even cut matroid which contains as a minor a non-degenerate 3-connected matroid \(N\). Then the number of equivalence classes of \(M\) is at most twice the number of equivalence classes of \(N\).

This result implies, in particular, that every 3-connected even cut matroid containing \(R_{10}\) as a minor has, up to equivalence, at most 20 representations. We will strengthen this result in Section 1.3.2.

We will show that degenerate even cut matroids are “close” to being co-graphic matroids. We require a number of definitions to formalize this notion.

Let \(N\) and \(M\) be matroids where \(E(N) = E(M)\). Then \(N\) is a lift of \(M\) if, for some matroid \(M'\) where \(E(M') = E(M) \cup \{\Omega\}\), \(M = M'/\Omega\) and \(N = M' \setminus \Omega\). If \(N\) is a lift of \(M\) then \(M\) is a projection of \(N\). Lifts and projections were introduced in \([2]\). Every even cut matroid \(M\) is a lift of a co-graphic matroid; indeed, for any representation \((G, T)\) of \(M\) we may construct \((G', T')\) by adding an odd bridge \(\Omega\). Then \(\text{ecut}(G', T')/\Omega\) is a co-graphic matroid. The following result shows that degenerate even cut matroids are projections of co-graphic matroids.
Remark 6. Let \((H,S)\) be a graft.

(1) If \(|S| \leq 2\), then \(\text{ecut}(H,S)\) is a co-graphic matroid.

(2) If \(|S| = 4\), then \(\text{ecut}(H,S)\) is a projection of a co-graphic matroid.

Proof. (1) The result is obvious if \(S = \emptyset\). Now suppose that \(|S| = 2\). Let \(G\) be obtained from \(H\) by identifying the two vertices in \(S\). Then \(\text{cut}(G) = \text{ecut}(H,S)\).

(2) Suppose that \(|S| = 4\). Let \(G\) be obtained from \(H\) by adding an edge \(\Omega = (s_1, s_2)\), for distinct \(s_1, s_2 \in S\). Let \(M' = \text{ecut}(G,T)\). Then by construction \((G,T) \setminus \Omega = (H,S)\), hence \(M'/\Omega = M\). By definition of the minor operations on grafts, \((G,T)/\Omega\) has two terminals. Therefore, by (1), \(\text{cut}((G,T)/\Omega) = M' \setminus \Omega\) is a co-graphic matroid.

1.3.2 Substantial minors

Consider a graft \((G,T)\) and suppose that there exist graphs \(G_1\) and \(G_2\) equivalent to \(G\) and paths \(P_1\) and \(P_2\) in \(G_1\) and \(G_2\) respectively, such that \(T = \text{odd}(G[P_1 \triangle P_2])\). We call the pair \((G_1, P_1)\) and \((G_2, P_2)\) a reaching pair for \((G,T)\). If \((G,T)\) is degenerate, then there exist (possibly empty) paths \(P_1\) and \(P_2\) in \(G\) such that \(T = \text{odd}(G[P_1 \triangle P_2])\); hence \((G,P_1), (G,P_2)\) is a reaching pair for \((G,T)\). It follows that having no reaching pair is a stronger property than being non-degenerate.

An even cut matroid is substantial if none of its representations has a reaching pair. Hence, if an even cut matroid is degenerate, it is not substantial. In particular, substantial matroids are not co-graphic. We will see (in Remark 13) that not being substantial is also a minor closed property.

It follows immediately from the definition that an even cut matroid \(M\) is substantial if, for every representation \((G,T)\) of \(M\), the graph \(G\) is 3-connected and \(|T| \geq 6\). Recall that every representation of \(R_{10}\) is isomorphic to the graft in Figure 3 and the representations of \(R_{10}\) partition into 10 equivalence classes. The graft obtained by contracting the bridge in the graft in Figure 3 is 3-connected and has six terminals, hence it has no reaching pair; it follows that no representation of \(R_{10}\) has a reaching pair, hence \(R_{10}\) is a substantial even cut matroid.

We are now ready to present the second main result of the paper.

Theorem 7. Let \(M\) be a connected even cut matroid which contains as a minor a connected matroid \(N\) that is substantial. Then the number of equivalence classes of \(M\) is at most the number of equivalence classes of \(N\).

This result implies, in particular, that every connected even cut matroid containing \(R_{10}\) as a minor has, up to equivalence, at most 10 representations.
1.4 Related results and motivation

Even cut matroids are a natural class of matroids to study as they are the smallest minor closed class of binary matroids which contains all single element co-extensions of co-graphic matroids. Robertson and Seymour [8] proved that, for every infinite set of graphs, one of its members is isomorphic to a minor of another. Geelen, Gerards, and Whittle announced that an analogous result holds for binary matroids. Hence, any minor closed class of binary matroids can be characterized by a finite set of excluded minors. In particular, this is the case for even cut matroids. Tutte [11] gave an explicit description of the excluded minors for the class of graphic matroids. By duality, this immediately provides an explicit description of the excluded minors for the class of co-graphic matroids. Tutte also gave a polynomial time algorithm to check if a binary matroid (given by its 0,1 matrix representation) is graphic [12].

No explicit description of the excluded minors is known for even cut matroids and we do not know how to recognize efficiently whether a given binary matroid is an even cut matroid. The difficulty for both problems lies with the fact that we do not have a sufficient understanding of the representations of even cut matroids. Theorems 5 and 7 are a first step toward a better understanding of this problem. Eventually, we wish to extend the aforementioned theorems so as to have a compact description of the representations of arbitrary even cut matroids. We believe that there exists a constant $k$ such that every even cut matroid with more than $k$ inequivalent representations is constructed in a way analogous to that of the Shuffle described in Section 1.2.

1.5 Organization of the paper

Section 2 introduces generalizations of Theorems 5 and 7. Sections 2.2 and 2.3 contain the proof of these theorems, modulo the exclusion of three key lemmas (namely Lemmas 12, 14 and 15). Lemma 12 is proved in Section 2.4. Lemma 14 is proved in Section 3. In Section 4 we give a characterization of special pairs of representations of the same matroid. This characterization is used to prove Lemma 15, in Section 5.

2 The proofs (modulo the exclusion of several lemmas)

If $N$ is a minor of a matroid $M$ then $M$ is a major of $N$. Consider an even cut matroid $M$ with a representation $(G,T)$. Let $I$ and $J$ be disjoint subsets of $E(M)$ and let $N := M \setminus I/J$. Let $(H,S) := (G,T)/I \setminus J$. It follows from Remark 4 that $(H,S)$ is a representation of $N$. We say that $(G,T)$ is an
extension to $M$ of the representation $(H, S)$ of $N$, or alternatively that $(H, S)$ extends to $M$.

The following result implies Theorem 5.

**Theorem 8.** Let $N$ be a 3-connected non-degenerate even cut matroid. Let $M$ be a 3-connected major of $N$. For every equivalence class $\mathcal{F}$ of $N$, the set of extensions of $\mathcal{F}$ to $M$ is the union of at most two equivalence classes.

The "at most two" in the previous theorem is best possible. Consider for instance the example in Figure 2. Observe that the grafts obtained from (a) and (b) by deleting the edge $\Omega$ are equivalent. However, (a) and (b) are not equivalent.

The following result implies Theorem 7.

**Theorem 9.** Let $N$ be a connected substantial even cut matroid. Let $M$ be a connected major of $N$. For every equivalence class $\mathcal{F}$ of $N$, the set of extensions of $\mathcal{F}$ to $M$ is contained in one equivalence class.

The proofs of Theorems 8 and 9 are constructive. Thus, given a description of the inequivalent representations of $N$, it is possible to construct the set all inequivalent representations of $M$.

### 2.1 Definitions

First an easy observation.

**Remark 10.** If $G_1$ and $G_2$ are equivalent graphs and $I$ and $J$ are disjoint subsets of $E(G)$, then $G_1 \setminus I/J$ and $G_2 \setminus I/J$ are equivalent.

**Proof.** Since $G_1$ and $G_2$ are equivalent, $\text{cut}(G_1) = \text{cut}(G_2)$. Hence, $\text{cut}(G_1)/I \setminus J = \text{cut}(G_2)/I \setminus J$. As the minor operations on graphs and matroids commute, we have, $\text{cut}(G_1 \setminus I/J) = \text{cut}(G_2 \setminus I/J)$. The result now follows from Theorem 1. \( \square \)

Consider a matroid $M$ and let $N := M \setminus I/J$ be a minor of $M$. If $J = \emptyset$ and $|I| = 1$ then $M$ is a column major of $N$. If $I = \emptyset$ and $|J| = 1$ then $M$ is a row major of $N$. A set $\mathcal{F}$ of representations of an even cut matroid is closed under equivalence if, for every $(H, S) \in \mathcal{F}$ and $(H', S')$ equivalent to $(H, S)$, we have that $(H', S') \in \mathcal{F}$.

**Remark 11.** Let $\mathcal{F}$ be a set of representations of an even cut matroid $N$ and let $M$ be a major of $N$. If $\mathcal{F}$ is closed under equivalence, then so is the set $\mathcal{F}'$ of extensions of $\mathcal{F}$ to $M$.  

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Proof. Let \((G,T) \in \mathcal{F}'\) and let \((G',T')\) be equivalent to \((G,T)\). We have \(N = M \setminus D/C\) for some \(D,C \subseteq E(M)\). Moreover, \((H,S) := (G,T)/D \setminus C\) and \((H',S') := (G',T')/D \setminus C\) are equivalent (see Remark 10). Since \((G,T) \in \mathcal{F}'\), we have \((H,S) \in \mathcal{F}\). As \(\mathcal{F}\) is closed under equivalence, \((H',S') \in \mathcal{F}'\). Hence, by definition, \((G',T') \in \mathcal{F}'\).

Let \(\mathcal{F}\) be an equivalence class of an even cut matroid \(N\). We say that \(\mathcal{F}\) is row stable (resp. column stable) if for all row (resp. column) majors \(M\) of \(N\), where \(M\) has no loop and no co-loop and \(M\) is not co-graphic, the set of extensions of \(\mathcal{F}\) to \(M\) is an equivalence class.

2.2 A sketch of the proof of Theorem 9

We postpone the proof of the following result until Section 2.4.

**Lemma 12.** Every equivalence class of an even cut matroid is column stable.

The following implies that if a matroid is not substantial then neither are any of its minors.

**Remark 13.** If \((G,T)\) has a reaching pair, so does every minor \((H,S)\) of \((G,T)\).

**Proof.** Since \((G,T)\) has a reaching pair, there exists, for \(i = 1, 2\), a graph \(G_i\) equivalent to \(G\) and a path \(P_i\) in \(G_i\) such that \(T = V_{\text{odd}}(G[P_1 \triangle P_2])\). By induction, it suffices to prove the statement for the cases \((H,S) = (G,T) \setminus e\) and \((H,S) = (G,T)/e\), for some \(e \in E(G)\).

First, suppose that \((H,S) = (G,T)/e\). For \(i = 1, 2\), define \(H_i := G_i/e\) and let \(Q_i\) be the (possibly empty) path in \(H_i\) obtained by removing all the cycles from \(H_i[P_i \setminus e]\). As \(H\) and \(H_i\) are equivalent, every cycle of \(H_i\) is a cycle of \(H\), hence \(V_{\text{odd}}(H[P_i \setminus e]) = V_{\text{odd}}(H(Q_i))\). As \(P_i \triangle P_2\) is a \(T\)-join of \(G\), \((P_1 \triangle P_2) \setminus e\) is an \(S\)-join of \(H\). Hence \(S = V_{\text{odd}}(H(Q_1 \triangle Q_2))\), and the statement follows.

Now suppose that \((H,S) = (G,T) \setminus e\). If \(e\) is an odd bridge of \(G\), then \(S\) is empty and the statement is trivially true (taking as reaching pair \((H,\emptyset), (H,\emptyset)\)). If \(e\) is not an odd bridge of \((G,T)\), then \(S = T\).

If \(e\) is an even bridge of \((G,T)\), then \((G,T)/e\) is equivalent to \((G,T) \setminus e\) (joining the two components of \(G \setminus e\) on the endpoints of \(e\) is a Whitney-flip). It follows (by the first part of the proof) that \((G,T)\) has a reaching pair. Thus we may assume that \(e\) is not a bridge of \(G\).

For \(i = 1, 2\), let \(v_i\) and \(w_i\) be the ends of \(P_i\) in \(G_i\) and \(H_i := G_i \setminus e\). Let \(Q_i\) be a \((v_i,w_i)\)-path in \(H_i\) (\(Q_i\) exists, as \(e\) is not a bridge of \(G\), hence \(e\) is not a bridge of \(G_i\)). Then \(P_i \triangle Q_i\) is a cycle of \(G_i\), hence a cycle of \(G\). It follows that \(V_{\text{odd}}(G[P_i]) = V_{\text{odd}}(G(Q_i))\), for \(i = 1, 2\). Therefore \(T = V_{\text{odd}}(H(Q_1 \triangle Q_2))\) and \((H_1,Q_1), (H_2,Q_2)\) is a reaching pair for \((H,T)\).
We say that an equivalence class $F$ has no reaching pair if none of the grafts in $F$ have a reaching pair. Note that we could replace “none” by “any” in the previous definition, as by definition, if a graft has a reaching pair, then so does every equivalent graft. We postpone the proof of the following result until Section 3.

**Lemma 14.** Equivalence classes without reaching pairs are row stable.

**Proof of Theorem 9.** Let $N$ be a connected non-degenerate even cut matroid. Let $M$ be a connected major of $N$. Then (see [1, 9]) there is a sequence of connected matroids $N_1, \ldots, N_k$, where $N = N_1$, $M = N_k$ and, for all $i \in [k-1]$, $N_{i+1}$ is a row or column major of $N_i$. In particular, $N_i$ has no loops or co-loops, for any $i \in [k]$. Since $N_1$ is substantial, it is not co-graphic, hence neither are $N_2, \ldots, N_k$. Let $F$ be an equivalence class of $N$ which extends to $M$ and, for every $j \in [k]$, define $F_j$ to be the set of extensions of $F$ to $N_j$. It suffices to show that, for all $j \in [k]$, $F_j$ is an equivalence class. Let us proceed by induction. As $N_1 = N$, the result holds for $j = 1$. Suppose that the result holds for $j \in [k-1]$. By Remark 13, $F_j$ does not have a reaching pair. Therefore, by Lemma 12 and Lemma 14, $F_j$ is row and column stable. It follows that $F_{j+1}$ is an equivalence class.

2.3 A sketch of the proof of Theorem 8

We say that an equivalence class of an even cut matroid is non-degenerate if all grafts in the class are non-degenerate. We postpone the proof of the following result until Section 5.

**Lemma 15.** Let $N$ be an even cut matroid and let $F$ be an equivalence class of $N$ that is non-degenerate. Let $M$ be a row major of $N$ with no loops or co-loops. Suppose that $N$ and $M$ are 3-connected and suppose that the set $F'$ of extensions of $F$ to $M$ is non-empty. Then $F'$ is either an equivalence class or the union of two equivalence classes $F_1$ and $F_2$ without reaching pairs.

**Proof of Theorem 8.** Let $N$ be a 3-connected non-degenerate even cut matroid. Let $M$ be a 3-connected major of $N$. It follows (by [10]) that there is a sequence of 3-connected matroids $N_1, \ldots, N_k$, where $N = N_1$, $M = N_k$ and, for every $i \in [k-1]$, $N_{i+1}$ is a row or column major of $N_i$. In particular, $N_i$ has no loops or co-loops for any $i \in [k]$. Since $N_1$ is non-degenerate, it is not co-graphic, hence neither are $N_2, \ldots, N_k$. Let $F$ be an equivalence class of $N$ that extends to $M$. For every $j \in [k]$, define $F_j$ to be the set of extensions of $F$ to $N_j$. It suffices to show that, for all $j \in [k]$, $F_j$ is either

(a) an equivalence class, or

(b) the union of two equivalence classes without reaching pairs.
Let us proceed by induction. As $N_1 = N$, the result holds for $j = 1$. Suppose that the result holds for $j \in [k-1]$. Consider the case where $N_{j+1}$ is a column major of $N_j$. If (a) holds for $F_j$, then Lemma 12 implies that (a) holds for $F_{j+1}$. If (b) holds for $F_j$, then Lemma 12 and Remark 13 imply that either (a) or (b) holds for $F_{j+1}$. Consider the case where $N_{j+1}$ is a row major of $N_j$. If (a) holds for $F_j$, then Lemma 15 implies that either (a) or (b) holds for $F_{j+1}$. If (b) holds for $F_j$, then Lemma 14 implies that either of (a) or (b) holds for $F_{j+1}$.

\[ \square \]

2.4 Proof of Lemma 12

The next result, proved in [3], is an easy consequence of Theorem 1.

**Remark 16.** Suppose that $\text{ecut}(G_1, T_1) = \text{ecut}(G_2, T_2)$. If any odd cut of $(G_1, T_1)$ is a cut of $G_2$, then $G_1$ and $G_2$ are equivalent.

We say that two grafts $(G_1, T_1)$ and $(G_2, T_2)$ are *siblings* if $\text{ecut}(G_1, T_1) = \text{ecut}(G_2, T_2)$ and the graphs $G_1$ and $G_2$ are not equivalent. The following result is proved in [4]. We report the proof here for completeness.

**Lemma 17.** Let $(G_1, T_1)$ and $(G_2, T_2)$ be graft siblings and let $\Omega \in E(G_1)$. For $i = 1, 2$, let $(H_i, S_i) := (G_i, T_i)/\Omega$. Suppose that $(H_1, S_1)$ and $(H_2, S_2)$ are equivalent. Then, for $i = 1, 2$, either $\Omega$ is a loop of $G_i$ or $|T_i| = 2$ and $T_i$ are the ends of $\Omega$ in $G_i$. In particular, $\Omega$ is a co-loop of $\text{ecut}(G_1, T_1)$.

**Proof.** For $i = 1, 2$, denote by $v_i$ and $w_i$ the endpoints of edge $\Omega$ in $G_i$. We prove the statement for $i = 1$. Remark 16 implies that no odd cut of $(G_1, T_1)$ is a cut of $G_2$. Since $H_1$ and $H_2$ are equivalent, $\text{cut}(H_1) = \text{cut}(H_2)$. It follows that all odd cuts of $(G_1, T_1)$ use $\Omega$. Hence, $T_1 \subseteq \{v_1, w_1\}$. Similarly, we have that $T_2 \subseteq \{v_2, w_2\}$. If $\Omega$ is a loop of $G_1$, we are done. Suppose otherwise. If $T_1 = \emptyset$, then there exists an even cut $D$ of $(G_1, T_1)$ using $\Omega$; hence $\Omega$ is not a loop of $G_2$ and $T_2 \neq \{v_2, w_2\}$. But then $T_1 = T_2 = \emptyset$ and $\text{cut}(G_1) = \text{cut}(G_2)$ and it follows by Theorem 1 that $G_1$ and $G_2$ are equivalent, a contradiction. We conclude that $T_1 = \{v_1, w_1\}$, completing the proof.

\[ \square \]

**Proof of Lemma 12.** Let $F$ be an equivalence class of an even cut matroid $N$. Let $M$ be a column major of $N$, i.e. for some $\Omega \in E(M)$, $N = M \setminus \Omega$. Let $F'$ be the set of all extensions of $F$ to $M$. By the definition of column extensions we may assume that $M$ has no loops and co-loops. We need to show that $F'$ is an equivalence class. For otherwise there exist siblings $(G_1, T_1), (G_2, T_2) \in F'$. For $i = 1, 2$, let $(H_i, S_i) := (G_i, T_i)/\Omega$. Then $(H_1, S_1), (H_2, S_2) \in F$. In particular, $(H_1, S_1)$ and $(H_2, S_2)$ are equivalent. Hence, by Lemma 17, $\Omega$ is a loop or co-loop of $\text{ecut}(G_1, T_1)$, a contradiction.

\[ \square \]

It remains to prove Lemma 14 and 15. Lemma 14 (resp. Lemma 15) is proved in Section 3 (resp. 5).
3 Row extensions and reaching pairs

Before we proceed with the proof of Lemma 14 we establish some preliminaries in Sections 3.1 and 3.2.

3.1 Even cycle matroids

Given a graph $G$, we denote by cycle$(G)$ the set of all cycles of $G$. Since the cycles of $G$ correspond to the cycles of the graphic matroid of $G$, we identify cycle$(G)$ with that matroid and say that $G$ is a representation of that matroid. A signed graph is a pair $(G, \Sigma)$ where $\Sigma \subseteq E(G)$. We call $\Sigma$ a signature of $G$. A subset $B \subseteq E(G)$ is $\Sigma$-even (respectively $\Sigma$-odd) if $|B \cap \Sigma|$ is even (respectively odd). When there is no ambiguity we omit the prefix $\Sigma$ when referring to $\Sigma$-even and $\Sigma$-odd sets. Given a signed graph $(G, \Sigma)$, we denote by cycle$(G, \Sigma)$ the set of all cycles of $(G, \Sigma)$. It can be verified that cycle$(G, \Sigma)$ is the set of cycles of the even cycle matroid. We identify cycle$(G, \Sigma)$ with that matroid and say that $(G, \Sigma)$ is a representation of that matroid.

Given a signed graph $(G, \Sigma)$, we say that $\Sigma'$ is obtained from $\Sigma$ by a signature exchange if $\Sigma \triangle \Sigma'$ is a cut of $G$ (where $\triangle$ denotes symmetric difference). Every set $\Sigma'$ which may be obtained from $\Sigma$ by a signature exchange is a signature of $(G, \Sigma)$.

We will make repeated use of the following result (which was proved in [3]).

**Theorem 18.** Let $G_1$ and $G_2$ be inequivalent graphs on the same edge set.

1. Suppose that there exists a pair $\Sigma_1, \Sigma_2 \subseteq E(G_1)$ such that cycle$(G_1, \Sigma_1) = \text{cycle}(G_2, \Sigma_2)$. For $i = 1, 2$, if $(G_i, \Sigma_i)$ has no $\Sigma_i$-odd cycle, define $C_i := \emptyset$; otherwise let $C_i$ be an odd cycle of $(G_i, \Sigma_i)$. Let $T_i := V_{\text{odd}}(G_i|C_{3-i})$. Then $\text{ecut}(G_1, T_1) = \text{ecut}(G_2, T_2)$.

2. Suppose that there exists a pair $T_1 \subseteq V(G_1), T_2 \subseteq V(G_2)$ (where $|T_1|$ and $|T_2|$ are even) such that $\text{ecut}(G_1, T_1) = \text{ecut}(G_2, T_2)$. For $i = 1, 2$, if $T_i = \emptyset$ let $\Sigma_{3-i} = \emptyset$; otherwise let $t_i \in T_i$ and $\Sigma_{3-i} := \delta_{G_i}(t_i)$. Then cycle$(G_1, \Sigma_1) = \text{cycle}(G_2, \Sigma_2)$.

Moreover, if it exists, the pair $\Sigma_1, \Sigma_2$ is unique (up to signature exchange) and, if it exists, the pair $T_1, T_2$ is unique.

3.2 Clip siblings

We now introduce an operation on grafts which preserves even cuts. Consider a pair of equivalent graphs $H_1$ and $H_2$. Suppose that $P_i \subseteq E(H_i)$ is a path in $H_i$, for $i = 1, 2$. For $i = 1, 2$, let $G_i$ be the graph
obtained from $H_i$ by adding an edge $\Omega$ with endpoints the ends of $P_i$. Since $H_1$ and $H_2$ are equivalent, 
Theorem 1 implies that $\text{cut}(H_1) = \text{cut}(H_2)$ and hence that $\text{cycle}(H_1) = \text{cycle}(H_2)$. Thus, 

$$\text{ecycle}(G_1, \{\Omega\}) = \text{cycle}(H_1) = \text{cycle}(H_2) = \text{ecycle}(G_2, \{\Omega\}).$$

Theorem 18 implies that there exist $T_1 \subseteq V(G_1)$ and $T_2 \subseteq V(G_2)$ such that $\text{ecut}(G_1, T_1) = \text{ecut}(G_2, T_2)$. If $G_1$ and $G_2$ are inequivalent, such pair is unique (again by Theorem 18); in this case we say that the tuple $T = (H_1, P_1, H_2, P_2)$ is a clip-template and that $(G_1, T_1)$ and $(G_2, T_2)$ are clip siblings which arise from $T$. An explicit characterization of clip siblings is given in Section 4. Such characterization is needed to prove Lemma 15.

**Remark 19.** Let $T = (H_1, P_1, H_2, P_2)$ be a clip-template and let $(G_1, T_1)$ and $(G_2, T_2)$ be clip siblings that arise from $T$. Then, for $i = 1, 2$, we have $T_i = V_{\text{odd}}(G_i[P_3-i \cup \Omega])$.

**Proof.** As $P_i \cup \Omega$ is an odd cycle of $(G_i, \{\Omega\})$ for $i = 1, 2$, by Theorem 18 we have $T_i = V_{\text{odd}}(G_i[P_3-i \cup \Omega]) = V_{\text{odd}}(G_i[P_3-i]) \Delta V_{\text{odd}}(G_i[\Omega])$. As $\Omega$ and $P_i$ have the same ends in $G_i$, we have $V_{\text{odd}}(G_i[\Omega]) = V_{\text{odd}}(G_i[P_i])$. It follows that $T_i = V_{\text{odd}}(G_i[P_3-i]) \Delta V_{\text{odd}}(G_i[P_i]) = V_{\text{odd}}(G_i[P_1 \Delta P_2])$.

We illustrate this construction in Figure 4. The dashed lines indicate 2-separations. $H_2$ is obtained from $H_1$ by doing a Whitney flips on each of the 2-separations. White vertices represent terminal vertices $T_1$ in $G_1$ and $T_2$ in $G_2$.

![Figure 4: Clip siblings](image-url)
3.3 Proof Lemma 14

The following easy observation is the analogue to Remark 16 for the case of even cycle matroids (see [3] for a proof).

Remark 20. Suppose that $\text{ecycle}(G_1, \Sigma_1) = \text{ecycle}(G_2, \Sigma_2)$. If any odd cycle of $(G_1, \Sigma_1)$ is a cycle of $G_2$, then $G_1$ and $G_2$ are equivalent.

We define minor operations on signed graphs as follows. Let $(G, \Sigma)$ be a signed graph and let $e \in E(G)$. Then $(G, \Sigma) \setminus e$ is defined as $(G \setminus e, \Sigma \setminus \{e\})$. We define $(G, \Sigma)/e$ as $(G \setminus e, \emptyset)$ if $e$ is an odd loop of $(G, \Sigma)$ and as $(G \setminus e, \Sigma)$ if $e$ is an even loop of $(G, \Sigma)$; otherwise $(G, \Sigma)/e$ is equal to $(G/e, \Sigma')$, where $\Sigma'$ is any signature of $(G, \Sigma)$ which does not contain $e$. Observe that (see [7] for instance),

Remark 21. $\text{ecycle}(G, \Sigma) \setminus I/J = \text{ecycle}((G, \Sigma) \setminus I/J)$.

In particular, this implies that being an even cycle matroid is a minor closed property.

We say that two signed graphs $(G_1, \Sigma_1)$ and $(G_2, \Sigma_2)$ are siblings if $\text{ecycle}(G_1, \Sigma_1) = \text{ecycle}(G_2, \Sigma_2)$ and $G_1$ and $G_2$ are inequivalent. The following result is the analogue to Lemma 17 for even cycle matroids. This result is proved, for example, in [4]; we report the proof here for completeness.

Lemma 22. Consider even-cycle matroids $(G_1, \Sigma_1)$ and $(G_2, \Sigma_2)$ that are siblings and let $\Omega \in E(G_1)$. For $i = 1, 2$, let $(H_i, \Gamma_i) := (G_i, \Sigma_i) \setminus \Omega$. Suppose that $H_1$ and $H_2$ are equivalent. Then, for $i = 1, 2$, $\Omega$ is either a bridge of $G_i$ or a signature of $(G_i, \Sigma_i)$. In particular, $\Omega$ is a co-loop of $\text{ecycle}(G_1, \Sigma_1)$.

Proof. We prove the statement for $i = 1$. Remark 20 implies that no odd cycle of $(G_1, \Sigma_1)$ is a cycle of $G_2$. Since $H_1$ and $H_2$ are equivalent, $\text{cycle}(H_1) = \text{cycle}(H_2)$. It follows that all odd cycles of $(G_1, \Sigma_1)$ use $\Omega$. Hence, after possibly a signature exchange, $\Sigma_1 \subseteq \{\Omega\}$. Similarly, we may assume that $\Sigma_2 \subseteq \{\Omega\}$. If $\Omega$ is a bridge of $G_1$, we are done. Suppose otherwise. If $\Sigma_1 = \emptyset$, then there exists an even cycle $C$ of $(G_1, \Sigma_1)$ using $\Omega$; hence $\Omega$ is not a bridge of $G_2$ and $\Sigma_2 \neq \{\Omega\}$. But then $\Sigma_1 = \Sigma_2 = \emptyset$ and $\text{cycle}(G_1) = \text{cycle}(G_2)$, so $\text{cut}(G_1) = \text{cut}(G_2)$. It follows by Theorem 1 that $G_1$ and $G_2$ are equivalent, a contradiction. \hfill \Box

Lemma 23. Let $N$ be an even cut matroid that is not co-graphic and let $\mathcal{F}$ be an equivalence class of $N$. Let $M$ be a row major of $N$ with no loops or co-loops. Suppose that the set $\mathcal{F}'$ of extensions of $\mathcal{F}$ to $M$ is non-empty. Then $\mathcal{F}'$ is either an equivalence class or the union of two equivalence classes $\mathcal{F}_1$ and $\mathcal{F}_2$ and any $(G_1, T_1) \in \mathcal{F}_1$ and $(G_2, T_2) \in \mathcal{F}_2$ are clip siblings.
Proof. We may assume that \( \mathbb{F}' \) is not an equivalence class. Hence, there exist siblings \((G_1, T_1)\) and \((G_2, T_2)\) in \( \mathbb{F}' \). Let \( \Omega \) denote the unique element in \( E(M) - E(N) \). Then \((G_1, T_1) \setminus \Omega\) and \((G_2, T_2) \setminus \Omega\) are in \( \mathbb{F} \). By Theorem 18, there is a unique (up to resigning) pair of signatures \( \Sigma_1 \) and \( \Sigma_2 \) for \( G_1 \) and \( G_2 \) such that \( \text{ecycle}(G_1, \Sigma_1) = \text{ecycle}(G_2, \Sigma_2) \). For \( i = 1, 2 \), let \((H_i, \Gamma_i) = (G_i, \Sigma_i) \setminus \Omega \). As \( H_1 \) and \( H_2 \) are equivalent, Lemma 22 implies that, for \( i = 1, 2 \), either \( \Omega \) is a bridge of \( G_i \) or a signature of \((G_i, \Sigma_i)\).

If the latter case occurs for both \( i = 1 \) and \( i = 2 \), then \((G_1, T_1)\) and \((G_2, T_2)\) are clip siblings and we are done. Now suppose that \( \Omega \) is a bridge of \( G_i \), for \( i = 1 \) or \( i = 2 \). Then every cycle of \( G_i \) is a cycle of \( H_i \), hence a cycle of \( H_{3-i} \) (as \( H_1 \) and \( H_2 \) are equivalent). It follows that every cycle of \( G_i \) is a cycle of \( G_{3-i} \). By Remark 20, every cycle of \((G_i, \Sigma_i)\) is even. Therefore \( \Sigma'_i = \emptyset \) is a signature of \((G_i, \Sigma_i)\). By Theorem 18, \( T_{3-i} \) is empty and \( M \) is co-graphic, a contradiction.

It remains to show that \( \mathbb{F}' \) can be partitioned into at most two equivalence classes. Suppose, for a contradiction, that this is not the case. Then there exist, for \( i = 1, 2, 3 \), \((G_i, T_i) \in \mathbb{F}' \), where \( G_1, G_2 \) and \( G_3 \) are pairwise inequivalent. For \( i = 1, 2, 3 \), let \( \Sigma_i := \Omega \). It follows from the argument in the previous paragraph that \( \text{ecycle}(G_1, \Sigma_1) = \text{ecycle}(G_3, \Sigma_3) \). Similarly, we have \( \text{ecycle}(G_2, \Sigma_2) = \text{ecycle}(G_3, \Sigma_3) \).

For \( i = 1, 2 \), let \( C_i \) be an odd cycle of \((G_i, \Sigma_i)\); note that, by the definition of \( \Sigma_i \), we have \( \Omega \in C_i \) for \( i = 1, 2 \). Theorem 18 applied to the pair \( G_1, G_2 \) implies that \( T_3 = V_{\text{odd}}(C_1) \). Similarly, Theorem 18 applied to the pair \( G_2, G_3 \) implies that \( T_3 = V_{\text{odd}}(C_2) \). Therefore \( V_{\text{odd}}(G_3[C_1 \Delta C_2]) = T_3 \Delta T_3 = \emptyset \), i.e. \( C_1 \Delta C_2 \) is a cycle of \( G_3 \). Moreover \( \Omega \notin C_1 \Delta C_2 \), as \( \Omega \in C_1 \cap C_2 \). By definition \( \Sigma_3 = \{ \Omega \} \), so \( C_1 \Delta C_2 \) is an even cycle of \((G_3, \Sigma_3)\). As \( \text{ecycle}(G_1, \Sigma_1) = \text{ecycle}(G_3, \Sigma_3) \), the set \( C_1 \Delta C_2 \) is a cycle of \( G_1 \). Hence \( C_2 = (C_1 \Delta C_2) \Delta C_1 \) is a cycle of \( G_1 \). Remark 20 implies that \( G_1 \) and \( G_2 \) are equivalent, a contradiction. 

We are now ready for the main result of this section,

**Proof of Lemma 14.** Let \( \mathbb{F} \) be an equivalence class of an even cut matroid \( N \) without reaching pairs. Let \( M \) be a row major of \( N \) which is not co-graphic and has no loops or co-loops. Let \( \Omega \) be the unique element in \( E(M) - E(N) \). Suppose by contradiction that the set of extensions of \( \mathbb{F} \) to \( M \) is non-empty and is not an equivalence class. By Lemma 23 the set of extensions of \( \mathbb{F} \) to \( M \) is the union of two equivalence classes \( \mathbb{F}_1 \) and \( \mathbb{F}_2 \) and any \((G_1, T_1) \in \mathbb{F}_1 \) and \((G_2, T_2) \in \mathbb{F}_2 \) are clip siblings that arise from some template \( T = (H_1, P_1, H_2, P_2) \) where, for \( i = 1, 2 \), \( H_i = G_i \setminus \Omega \). Moreover, \((H_i, T_i) \in \mathbb{F}_i \), for \( i = 1, 2 \). Remark 19 states that \( T_i = V_{\text{odd}}(G_i[P_i \Delta P_2]) \), for \( i = 1, 2 \). Hence \( (H_i, T_i) \in \mathbb{F} \), for \( i = 1, 2 \). It follows that \((H_1, P_1)\) and \((H_2, P_2)\) form a reaching pair of \((H_1, T_1)\), a contradiction. 

\[ \square \]
4 A characterization of clip siblings

We only need to prove Lemma 15 to complete the paper. One ingredient will be Lemma 23. The other ingredient is a theorem that gives a structural characterization of clip siblings. Before we can state this theorem, however, we need to understand 3-connected even cut matroids.

4.1 Connectivity

Let $M$ be a matroid with rank function $r$. Given $X \subseteq E(M)$ we define $\lambda_M(X)$, the connectivity function of $M$, to be equal to $r(X) + r(\bar{X}) - r(E(M)) + 1$. The set $X$ is a $k$-separation of $M$ if $\min\{|X|, |\bar{X}|\} \geq k$ and $\lambda_M(X) = k$. $M$ is $k$-connected if it has no $r$-separations for any $r < k$. Let $G$ be a graph and let $X \subseteq E(G)$. The set $X$ is a $k$-separation of $G$ if $\min\{|X|, |\bar{X}|\} \geq k$, $|B_G(X)| = k$ and both $G[X]$ and $G[\bar{X}]$ are connected. A graph $G$ is $k$-connected if it has no $r$-separations for any $r < k$.

Given a separation $X$ of $G$, we define the interior of $X$ in $G$ to be $I_G(X) = V_G(X) - B_G(X)$. Given a graft $(G, T)$, we say that an edge $e$ of $G$ is a pin if $e$ is an odd bridge of $G$ incident to a vertex of degree one, which we call the head of the pin. By definition the head of a pin is a terminal. We denote by $\text{pin}(G, T)$ the set of pins of $(G, T)$.

Proposition 24. Suppose that $\text{ecut}(G, T)$ is 3-connected. Then:

1. $|\text{pin}(G, T)| \leq 1$;
2. $G/\text{pin}(G, T)$ is 2-connected;
3. if $G$ has a 2-separation $X$ then $T \cap I_G(X)$ and $T \cap I_G(\bar{X})$ are both non-empty.

To prove Proposition 24, we require a definition and a preliminary result. We say that $X$ is a $k$-$(i, j)$-separation of a graft $(G, T)$, where $i, j \in \{0, 1\}$, if the following hold:

- $X$ is a $k$-separation of $G$;
- $i = 0$ when $T \cap I_G(X)$ is empty and $i = 1$ otherwise;
- $j = 0$ when $T \cap I_G(\bar{X})$ is empty and $j = 1$ otherwise.

Lemma 25. Let $(G, T)$ be a graft, where $T$ is non-empty and $G$ is connected. Let $M := \text{ecut}(G, T)$. For every $k$-$(i, j)$-separation $X$ of $(G, T)$, we have $\lambda_M(X) = k + i + j - 1$. 

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Proof. Let \( N = \text{cut}(G) \) with rank function \( r \). As the dual of \( N \) is cycle\((G)\), we have:

\[
\lambda_N(X) = r(X) + r(\bar{X}) - r(E(N)) + 1 = k. \tag{*}
\]

As \( T \) is non-empty and \( G \) is connected, a basis for \( M \) consists of the complement of the edge set of a spanning tree plus an edge forming an odd cut with this set of edges. Hence, the rank of \( M \) is one larger than the rank of \( N \). To compute the rank of \( X \) for \( N \) we delete the elements \( \bar{X} \) of \( N \) (i.e. we contract the edges \( \bar{X} \) of \( G \)) and consider the size of the complement of a spanning forest. To compute the rank of \( X \) for \( M \) we contract the set of edges in \( \bar{X} \) (in \( G \)) and consider the size of the complement of a spanning forest plus possibly an edge forming an odd cut with this set of edges. Hence, if \( T \cap \mathcal{F}_G(X) \) (respectively \( T \cap \mathcal{F}_G(\bar{X}) \)) is non-empty, then the rank of \( X \) (respectively \( \bar{X} \)) in \( M \) is one more that in \( N \), otherwise the rank of \( X \) (respectively \( \bar{X} \)) is the same in both matroids. Thus the rank of \( X \) in \( M \) is \( r(X) + i \) and the rank of \( \bar{X} \) in \( M \) is \( r(\bar{X}) + j \). The result follows from (*). \hfill \Box

Proof of Proposition 24. Let \( M := \text{ecut}(G, T) \). We may assume that \( G \) is connected as we can identify vertices in distinct blocks of \( G \) without changing the even cut matroid. (Moreover, as we will prove that \( G/\text{pin}(G, T) \) is 2-connected, this implies that originally, \( G/\text{pin}(G, T) \) was connected.) As \( M \) is 3-connected, it has no loops, no co-loops and no parallel elements. We may assume that \( T \) is non-empty, for otherwise \( M = \text{cut}(G) \) and \( G \) is 3-connected. (1) There do not exist distinct pins \( e, f \) of \( G \), for otherwise \( \{e, f\} \) would be an even cut of \( G \) and \( e, f \) would be in parallel in \( M \). (2) Suppose that \( X \) is a 1-\((i, j)\)-separation of \((G, T)\). By Lemma 25, \( \lambda_M(X) = 1 + i + j - 1 \leq 2 \). As \( M \) is 3-connected, \( X \) is not a 2-separation; hence either \(|X| = 1 \) or \(|\bar{X}| = 1 \). The single element in \( X \) (or \( \bar{X} \)) is not a loop of \( G \), for otherwise it is a co-loop of \( M \). Hence \( X \) or \( \bar{X} \) is a pin of \( G \). (3) Suppose that \( X \) is a 2-\((i, j)\)-separation of \((G, T)\). As \( M \) is 3-connected, \( \lambda_M(X) \geq 3 \). By Lemma 25, \( 2 + i + j - 1 \geq 3 \), hence \( i = j = 1 \). \hfill \Box

4.2 Characterizing clip siblings

The following theorem provides a structural description of clip siblings. The proof of this result will be postponed until Section 4.3.

Theorem 26. Let \( M \) be a 3-connected even cut matroid with representations \((G_1, T_1)\) for \( i = 1, 2 \). Suppose that \((G_1, T_1)\) and \((G_2, T_2)\) are clip siblings arising from a clip-template \( T = (H_1, P_1, H_2, P_2) \), where \( \text{ecut}(H_1, T_1) \) is 3-connected and is not co-graphic. Then \((G_1, T_1)\) and \((G_2, T_2)\) are either basic siblings or nested siblings.

We need to define the terms “basic siblings” and “nested siblings”. We will define the more restrictive notion of basic and nested twins. We say that \((G_1, T_1)\) and \((G_2, T_2)\) are basic (respectively nested)
siblings if, for \( i = 1, 2 \), there exists \((G'_i, T'_i)\) equivalent to \((G_i, T_i)\) such that \((G'_1, T'_1)\) and \((G'_2, T'_2)\) are basic (respectively nested) twins.

We first need a number of definitions. By a sequence \((X_1, \ldots, X_k)\) we mean a family of sets \(\{X_1, \ldots, X_k\}\) where \(X_i\) precedes \(X_j\) when \(i < j\). We say that \(S = (X_1, \ldots, X_k)\) is a \(w\)-sequence of \(G\) if, for all \(i \in [k]\), \(X_i\) is a 2-separation of the graph obtained from \(G\) by performing Whitney-flips on \(X_1, \ldots, X_{i-1}\) in this order). We denote by \(W_{np}[G, S]\) the graph obtained from \(G\) by performing Whitney-flips on \(X_1, \ldots, X_k\) in this order). If \(S\) consists of a single set \(X\), then we write \(W_{np}[G, X]\) in lieu of \(W_{np}[G, S]\). If \(G\) and \(G'\) are equivalent graphs that are 2-connected, then \(G' = W_{np}[G, S]\) for some \(w\)-sequence \(S\) of \(G\). Consider a clip-template \((H_1, P_1, H_2, P_2)\) where \(H_2 = W_{np}[H_1, S]\) for some \(w\)-sequence \(S\) of \(H_1\). We slightly abuse terminology and call the tuple \((H_1, P_1, H_2, P_2, S)\) a clip-template.

We say that two sets \(X\) and \(Y\) are crossing if the sets \(X - Y, Y - X, X \cap Y, \bar{X} \cap \bar{Y}\) are non-empty. We say that a \(w\)-sequence is non-crossing if no two sets in the \(w\)-sequence are crossing. A sequence \((X_1, \ldots, X_k)\) is nested if \(X_i \subseteq X_{i+1}\) for all \(i \in [k-1]\). If a \(w\)-sequence is nested then it is non-crossing. If a \(w\)-sequence is non-crossing the graph obtained by performing the Whitney-flips on this sequence does not depend on the order in which the Whitney-flips are performed.

**Remark 27.** Suppose that \(H_2 = W_{np}[H_1, S]\) for some non-crossing \(w\)-sequence \(S\) of \(H_1\). Then for any sequence \(S'\) obtained by reordering \(S\) we have that, for \(i = 1, 2\), \(S'\) is a \(w\)-sequence of \(H_i\) and \(H_{3-i} = W_{np}[H_i, S']\).

We leave the proof of this last result as an exercise.

When considering a nested \(w\)-sequence \((X_1, \ldots, X_k)\), we will always assume that \(X_i\) and \(X_{i+1}\) have distinct boundaries, for all \(i \in [k-1]\). If this is not the case, we could just remove the sets \(X_i\) and \(X_{i+1}\) from the sequence.

### 4.2.1 Basic twins

Consider a clip-template \(T = (H_1, P_1, H_2, P_2, S)\). If \(S = \emptyset\) (that is \(H_1 = H_2\)) then \(T\) is a basic-template and \((G_1, T_1)\) and \((G_2, T_2)\) arising from \(T\) are basic twins. By Remark 19, \(T_i V_{odd}(H_1[P_1 \triangle P_2])\), for \(i = 1, 2\). As \(P_1\) and \(P_2\) are both paths in \(H_1\) and in \(H_2\), this implies that \(|T_1|, |T_2| \leq 4\). Therefore:

**Remark 28.** Basic twins are degenerate.

### 4.2.2 Nested twins

We say that a clip-template \(T = (H_1, P_1, H_2, P_2, S)\) is a nested-template if the following hold:
(A1) $S = (X_1, \ldots, X_k)$ is a nested w-sequence for $H_1$ (where $k \geq 1$);

(A2) for $i = 1, 2$, $P_i$ has one endpoint in $\mathcal{I}_H(X_1)$ and one endpoint in $\mathcal{I}_H(\bar{X}_k)$;

(A3) $P_1$ and $P_2$ have no common endpoint in both $H_1$ and $H_2$.

Moreover, if $e \in \text{pin}(H_i, T_i)$ for $i \in [2]$ and $p$ denotes the endpoint of $e$ that is not the head of $e$, then

(A4) $e \in P_1 \cup P_2$, $p \notin T$ and either $e \in X_1$, $p \notin \mathcal{B}_H(X_1)$ or $e \in \bar{X}_k$, $p \notin \mathcal{B}_H(\bar{X}_k)$.

In this case we say that the grafts $(G_1, T_1)$ and $(G_2, T_2)$ arising from $T$ are nested twins. Note that Remark 27 implies that (A1) is equivalent to the statement that $S$ is a nested w-sequence for $H_2$. An example of nested twins was given in Figure 4.

In the case of nested twins we can give an explicit characterization of the set of terminals. A caterpillar is a tree obtained by taking a path and adding edges which have exactly one endpoint in common with the path. Let $G$ be a graph and let $S = (X_1, \ldots, X_k)$ be a nested w-sequence for $G$. We denote by $\text{Cat}(G, S)$ the graph defined on the vertex set $\cup_{i=1}^{k} \mathcal{B}_G(X_i)$ with edge set $\{e_1, \ldots, e_k\}$, where the endpoints of $e_i$ are the vertices in $\mathcal{B}_G(X_i)$. Note that $\text{Cat}(G, S)$ is a vertex-disjoint union of caterpillars.

**Proposition 29.** Let $T = (H_1, P_1, H_2, P_2, S)$ be a nested-template where $S = (X_1, \ldots, X_k)$ and, for $i = 1, 2$, let $v_i$ denote the endpoint of $P_i$ in $\mathcal{I}_H(X_1)$ and $w_i$ denote the endpoint of $P_i$ in $\mathcal{I}_H(\bar{X}_k)$. Let $(G_1, T_1)$ and $(G_2, T_2)$ be the nested twins arising from $T$. Then, for $i = 1, 2$,

$$T_i = \{v_1, v_2, w_1, w_2\} \cup V_{\text{odd}}(\text{Cat}(H_i, S)).$$

Figure 5 illustrates this last proposition: grafts (a) and (b) are nested twins, say $(G_1, T_1)$ and $(G, T_2)$ respectively, arising from some clip-template $T = (H_1, P_1, H_2, P_2, S)$. White vertices are terminals, dashed lines represent the nested 2-separations $X \in S$. Then $\text{Cat}(H_1, S)$ is the graph in (a) where edges correspond to dashed lines. As indicated in the proposition, the odd degree vertices in that graph together with the vertices $v_1, v_2, w_1, w_2$ form the the set of terminals in that graft. Similarly for (b) the odd degree vertices of $\text{Cat}(H_2, S)$ together with $v_1, v_2, w_1, w_2$ form the set of terminals in that graft.

In the remainder of this section we prove Proposition 29. We first need to specify the way we relabel vertices in graphs when doing Whitney-flips. Let $G$ be a graph and let $X$ be a 2-separation with $\mathcal{B}_G(X) = \{u_1, u_2\}$, let $G'$ be obtained from $G$ by doing a Whitney-flip on $X$, i.e $G'$ is obtained by (i) identifying $u_1$ in $G[X]$ with vertex $u_2$ in $G[\bar{X}]$; and (ii) identifying $u_2$ in $G[X]$ with vertex $u_1$ in $G[\bar{X}]$. Throughout the remainder of the paper we will use the convention that the resulting vertex in (i) is
labeled \( u_1 \) and that the resulting vertex in (ii) is labeled \( u_2 \) (i.e. vertices of \( G' \) in \( \mathcal{B}_G(X) \) are labeled according to \( G[X] \)). Given a graft \((G, T)\) and a w-sequence \( S \) for \( G \), we denote by \( W_{\text{flip}}[(G, T), S] \) the graft \((G', T')\), where \( G' = W_{\text{flip}}[G, S] \) and \((G, T) \) and \((G', T') \) are equivalent.

**Remark 30.** Let \((H, T)\) be a graft and let \( X \) be a 2-separation of \( H \) with \( \mathcal{B}_H(X) = \{u_1, u_2\} \). If \( |\mathcal{I}_H(\bar{X}) \cap T| \) is odd, then for \((H', T') = W_{\text{flip}}[(H, T), X] \) we have \( T' = T \triangle \{u_1, u_2\} \).

**Proof.** Let \( J \) be a \( T \)-join of \( H \). Since \( H[J \cap \bar{X}] \) has an even number of vertices of odd degree, and since by hypothesis there are an odd number of such vertices in \( \mathcal{I}_H(\bar{X}) \), exactly of the following set has odd cardinality: \( \delta_H(u_1) \cap J \cap \bar{X} \) or \( \delta_H(u_2) \cap J \cap \bar{X} \). Thus \( J \) is a \( T \triangle \{u_1, u_2\} \)-join of \( H' \). □

**Lemma 31.** Let \( H \) be a graph and let \( S = (X_1, \ldots, X_k) \) be a nested w-sequence for \( H \). Let \( v \in \mathcal{I}_H(X_1) \), \( w \in \mathcal{I}_H(\bar{X}_k) \) and let \((H', T) = W_{\text{flip}}[(H, \{v, w\}), S] \). Then \( T = \{v, w\} \cup V_{\text{odd}}(\text{Cat}(H', S)) \).

**Proof.** For all \( r \in [k] \), let \( S_r = (X_1, \ldots, X_r) \) and let \((H_r, T_r) = W_{\text{flip}}[(H, \{v, w\}), S_r] \). By induction we will show that for all such \( r \) we have \( T_r = \{v, w\} \cup V_{\text{odd}}(\text{Cat}(H_r, S_r)) \). The result holds for \( r = 1 \) by Remark 30. Suppose now the result holds for \( r < k \). Let \( u_1, u_2 \) denote the vertices in \( \mathcal{B}_H(X_r) \). Then

\[
T_{r+1} = T_r \triangle \{u_1, u_2\}
\]

\[
= \{v, w\} \cup V_{\text{odd}}(\text{Cat}(H_r, S_r)) \triangle \{u_1, u_2\}
\]

\[
= \{v, w\} \cup V_{\text{odd}}(\text{Cat}(H_{r+1}, S_{r+1}))
\]
where the first equality follows by Remark 30 (as \( \{w\} = \mathcal{S}_{H_i}(\tilde{X}_i) \cap T \)), the second equality follows by induction, and the third equality follows from the fact that \( \text{Cat}(H_{r+1}, S_{r+1}) \) is obtained from \( \text{Cat}(H_r, S_r) \) by adding the edge \( u_1u_2 \).

\[ \square \]

**Proof of Proposition 29.** Remark 19 implies that \( T_i = V_{\text{odd}}(G_i[P_i \triangle P_i]) = V_{\text{odd}}(G_i[P_i]) \triangle V_{\text{odd}}(G_i[P_i]) \). Since \( P_i \) is a path of \( G_i \) with endpoints \( v_i, w_i \), \( V_{\text{odd}}(G_i[P_i]) = \{v_i, w_i\} \). Finally, Lemma 31 implies that \( V_{\text{odd}}(G_i[P_{3-i}]) = \{v_{3-i}, w_{3-i}\} \cup V_{\text{odd}}(\text{Cat}(H_i, S)) \) and the result follows by (A3) in the definition of nested templates.

\[ \square \]

### 4.3 Proof of Theorem 26

Let \( G \) be a graph and \( P \) a path in \( G \). We say that a Whitney-flip on a 2-separation \( X \) preserves \( P \) if \( P \) is a path of \( W_{\text{wp}}[G, X] \). Note that this occurs if and only if the endpoints of \( P \) are both in \( V_G(X) \) or both in \( V_G(\tilde{X}) \). We say that a \( w \)-sequence \( S \) of \( G \) preserves \( P \) if \( P \) is a path in \( W_{\text{wp}}[G, S] \).

**Proposition 32.** Let \( H_1 \) and \( H_2 \) be equivalent 2-connected graphs and let \( P \) be a path in \( H_1 \). Then there exists a graph \( H \) such that:

1. \( H = W_{\text{wp}}[H_1, S_1] \), for some \( w \)-sequence \( S_1 \) of \( H_1 \) which preserves \( P \), and
2. \( H_2 = W_{\text{wp}}[H, S_2] \), for some nested \( w \)-sequence \( S_2 \) of \( H \), where no \( X \in S_2 \) preserves \( P \) in \( H \).

See [7](Proposition 5.4) for a proof of this result.

**Lemma 33.** Let \( H_1 \) and \( H_2 \) be equivalent 2-connected graphs with respectively path \( P_1 \) and \( P_2 \). Then there exist graphs \( H_1' \) and \( H_2' \) and \( S \) that is a nested \( w \)-sequence of both \( H_1' \) and \( H_2' \) such that, for \( i = 1, 2 \),

1. \( H_1' \) is equivalent to \( H_i \) and \( P_i \) is a path of \( H_i' \), and
2. \( H_2' = W_{\text{wp}}[H_1', S] \) and no \( X \in S \) preserve path \( P_i \) of \( H_i' \).

**Proof.** Proposition 32 implies that there exists \( H_1' \) equivalent to \( H_1 \) where \( P_1 \) is a path of \( H_1' \) and a nested \( w \)-sequence \( S_2 \) of \( H_1' \) such that \( H_2 = W_{\text{wp}}[H_1', S_2] \) where no \( X \in S_2 \) preserves path \( P_1 \) of \( H_1' \). Because of Remark 27, we can partition \( S_2 \) into \( S \) and \( S' \) such that \( H_2' = W_{\text{wp}}[H_2, S'] \) where \( P_2 \) is a path of \( H_2' \) and \( H_1' = W_{\text{wp}}[H_2', S] \) where no \( X \in S \) preserves path \( P_2 \) of \( H_2' \). Because of Remark 27 we may assume that \( S \) is nested, and it is a \( w \)-sequence of both \( H_1' \) and \( H_2' \).

\[ \square \]

**Proof of Theorem 26.** Let first consider the case where \( (H_i, T_i) \) for \( i = 1, 2 \) have no pins. It follows from Proposition 24 that \( H_1 \) and \( H_2 \) are 2-connected. Hence, \( H_1, H_2, P_1, P_2 \) satisfy the hypothesis of
Lemma 33. Let $H_1', H_2'$ and $S$ be obtained as in that lemma. Then properties (1) and (2) of the lemma imply that $T' = (H_1', P_1, H_2', P_2, S)$ is a clip-template. Let $(G_1', T_1'), (G_2', T_2')$ be the clip siblings arising from $T'$. Claim 1 and Claim 2 will imply that $(G_1, T_1)$ and $(G_2, T_2)$ are basic siblings or nested siblings.

Claim 1. For $i = 1, 2$, $(G_i, T_i)$ and $(G_i', T_i')$ are equivalent.

Proof. Let $i \in [2]$. Let $\Omega$ denote the edge in $E(G_i) - E(H_i)$. By construction,

\[
\text{cycle}(G_i) = \text{span}(\text{cycle}(H_i) \cup \{P_i \cup \{\Omega}\})
\]

\[
\text{cycle}(G_i') = \text{span}(\text{cycle}(H_i') \cup \{P_i \cup \{\Omega}\})
\]

By Theorem 1, $\text{cut}(H_i) = \text{cut}(H_i')$ or equivalently, $\text{cycle}(H_i) = \text{cycle}(H_i')$. It follows that $\text{cycle}(G_i) = \text{cycle}(G_i')$, or equivalently, $\text{cut}(G_i) = \text{cut}(G_i')$. Hence, by Theorem 1, $G_i$ and $G_i'$ are equivalent. Since $G_i$ and $G_i'$ are equivalent, for some set of terminals $R_i$, the grafts $(G_i, R_i)$ and $(G_i', T_i')$ are equivalent. Since $(G_1', T_1')$ and $(G_2', T_2')$ are clip siblings, $\text{cut}(G_1', T_1') = \text{cut}(G_2', T_2')$. Hence, $\text{cut}(G_1, R_1) = \text{cut}(G_2, R_2)$. Since $(G_1, T_1)$ and $(G_2, T_2)$ are clip siblings, $\text{cut}(G_1, T_1) = \text{cut}(G_2, T_2)$. It follows from Theorem 18 that $T_i = R_i$ for $i = 1, 2$. \hfill \diamond \\

Claim 2. $T' = (H_1', P_1, H_2', P_2, S)$ is either a basic-template or a nested-template.

Proof. If $S = \emptyset$ then by definition $T'$ is a basic-template. Thus we may assume that $S = (X_1, \ldots, X_k)$ for some $k \geq 1$. Property (2) of Lemma 33 implies that, for $i = 1, 2$, $P_i$ has one endpoint in $\mathcal{I}_H(X_1)$ and the other in $\mathcal{I}_H(\bar{X}_k)$. Thus properties (A1) and (A2) of nested-templates are satisfied. Suppose for a contradiction that (A3) does not hold, i.e. $P_1$ and $P_2$ have a common endpoint in $H_i$ for some $i \in [2]$. Up to relabeling we may assume that $P_1$ and $P_2$ have the same endpoint $v \in \mathcal{I}_H(X_1)$. By Remark 19, $T_1' = V_{\text{odd}}(G_1'[P_1 \Delta P_2])$, hence $v \notin T_1'$. It follows from Proposition 24 that $X_1$ is a 2-separation of $\text{cut}(H_1', T_1')$. By Claim 1, $\text{cut}(H_1', T_1') = \text{cut}(H_1, T_1)$, a contradiction as this matroid is 3-connected. As by hypothesis $(H_i, T_i)$ has no pins for $i = 1, 2$, property (A4) is vacuously true. \hfill \diamond \\

It remains to consider the case where $(H_1, T_1)$ and thus $(H_2, T_2)$ has a pin $e$.

Claim 3. We may assume $e \in P_1 - P_2$ and $P_1 \neq \{e\}$.

Proof. Let $h$ denote the head of the pin $e$ in $H_1$. Suppose for a contradiction that $e \in P_1 \cap P_2$ or that $e \notin P_1 \cup P_2$. By Remark 19, $T_1 = V_{\text{odd}}(G_1'[P_1 \Delta P_2])$. Thus $h \notin T_1$, a contradiction as $e$ is a pin of $(H_1, T_1)$. Thus we may assume that $e \in P_1 - P_2$. Suppose for a contradiction that $P_1 = \{e\}$. Remark 19 implies that $T_2 = V_{\text{odd}}(G_2[P_1 \Delta P_2])$. Hence, the only terminals in $(H_2, T_2)$ are the endpoints of path
\(P_2\) and the endpoints of \(P_1 = \{e\}\). It follows that the graft obtained from \((H_2, T_2)\) by moving the pin \(e\) to an endpoint of \(P_2\) has exactly two terminals. Hence, by Remark 6, \(\text{ecut}(H_2, T_2)\) is co-graphic, contradicting the hypothesis.

\[\Box\]

For \(i = 1, 2\) let \((\hat{H}_i, \hat{T}_i) = (H_i, T_i)/e\) and let \(\hat{P}_1 = P_1 - \{e\}\). Claim 3 implies that \(\hat{T} = (\hat{H}_1, \hat{P}_1, \hat{H}_2, P_2)\) is a clip-template. Proposition 32 implies that \(\hat{H}_1\) and \(\hat{H}_2\) are 2-connected. Thus we can now apply the previous argument to \(\hat{T}\). At the end we uncontract the pin \(e\). It suffices to observe that as property (A2) holds before uncontracting \(e\), property (A4) will hold after uncontracting \(e\).

\[\square\]

5 \ Row extensions and non degenerate matroids

The goal of this section is to prove Lemma 15.

5.1 The proof (modulo the exclusion of one lemma)

A graft \((G, T)\) is nice if there exist an edge \(\Omega\) that is not a bridge of \(G\) and a nested \(w\)-sequence \(S = (X_1, \ldots, X_k)\) of \(H = G \setminus \Omega\) such that the following hold:

- (B1) there exist \(v_1, v_2 \in \mathcal{I}_H(X_1) \cap T\) and \(w_1, w_2 \in \mathcal{I}_H(X_k) \cap T\);
- (B2) \(T = \{v_1, v_2, w_1, w_2\} \cup T_e\) where \(T_e \subseteq \bigcup_{i=1}^k \mathcal{I}_H(X_i)\) and \(v_1, v_2, w_1, w_2\) are all distinct;
- (B3) \(\Omega\) has endpoints \(v_1, w_1\);
- (B4) \((H, T) = (G, T) \setminus \Omega\) is non-degenerate;
- (B5) \(H\) is 2-connected.

Lemma 34. Suppose that \((G_1, T_1)\) and \((G_2, T_2)\) are siblings arising from a nested-template \(T = (H_1, P_1, H_2, P_2, S)\), where \((H_1, T_1)\) is non-degenerate and \(\text{ecut}(H_1, T_1)\) is 3-connected and not co-graphic. Then, for \(i = 1, 2\), \((G_i, T_i)\) contains as a minor a nice graft.

Proof.\ We may assume \(i = 1\). Let \(\Omega\) denote the element in \(E(G_1) - E(H_1)\). \(T\) satisfies properties (A1)-(A4) of nested templates (see Section 4.2.2). (A1) states that \(S = (X_1, \ldots, X_k)\) is a nested \(w\)-sequence for \(H_1\) \((k \geq 1)\). (A2) implies that for \(i = 1, 2\), path \(P_i\) has an endpoint \(v_i \in \mathcal{I}_H(X_1)\) and an endpoint \(w_i \in \mathcal{I}_H(X_k)\). By (A3), \(v_1 \neq v_2\) and \(w_1 \neq w_2\). Thus \(v_1, v_2, w_1, w_2\) are all distinct. Proposition 29 implies that \(T_1 = \{v_1, v_2, w_1, w_2\} \cup V_{\text{odd}}(\text{Cat}(H_1, S))\). Hence, (B1) and (B2) hold for \((H_1, T_1)\). Moreover, since
the endpoints of $P_1$ are $v_1, w_1$, (B3) holds as well. By hypothesis $(H_1, T_1)$ is non-degenerate, i.e. (B4) holds for $(H_1, T_1)$ as well.

Proposition 24 implies that $H_1$ is 2-connected except for a possible pin $e$ of $(H_1, T_1)$. If there is no pin $e$, then $H_1$ is 2-connected and $(H_1, T_1)$ satisfies (B1)-(B5), i.e. is the required nice graft. Thus we may assume there exists a pin $e$. Clearly, $H_1/e$ is 2-connected. Hence, to show that $(H_1, T_1)/e$ is a nice graft, it suffices to verify that (B1)-(B4) hold for $(H_1, T_1)/e$. (B1)-(B3) follow from the fact that $(H_1, T_1)$ satisfy (B1)-(B3) and the fact that $T$ satisfies (A4). Finally, it can be readily checked that contracting a pin in a graft preserves non-degeneracy, i.e. $(H_1, T_1)/e$ satisfies (B4).

Let $(G, T)$ be a graft and let $X$ be a 2-separation of $G$ where $\mathcal{R}_G(X) = \{u_1, u_2\}$. We say that $X$ is simple if $\mathcal{R}_G(X) = \{v\}$, $v \in T$ and either $X = \{u_1v, u_2v\}$ or $X = \{u_1v, u_2v, u_1u_2\}$. A graft $(G, T)$ is nearly 3-connected if $G$ is 2-connected and for every 2-separation at least one of $X$ or $\bar{X}$ is simple.

**Remark 35.** In a nearly 3-connected non-degenerate graft $(G, T)$ no pair of 2-separations $X_1$ and $X_2$ cross.

**Proof.** Suppose for a contradiction that $X_1$ and $X_2$ cross. Then there exist series edges $e, f, g$ with $X_1 = \{e, f\}$ and $X_2 = \{f, g\}$. Since $(G, T)$ is nearly 3-connected $E(G) - X_1 - X_2$ is simple. It follows that $(G, T)$ is degenerate, a contradiction.

**Lemma 36.** Nearly 3-connected grafts with a reaching pair are degenerate.

**Proof.** Suppose that a nearly 3-connected graft $(G, T)$ has a reaching pair $(G_1, P_1)$ and $(G_2, P_2)$. By Remark 35 we may assume that 2-separations of $G$ are non-crossing. In particular, simple separations of $G$ are simple separations of $G_1$ and $G_2$ (and vice versa) and $G_1$ and $G_2$ are nearly 3-connected.

Hence, $G_1$ and $G_2$ are 2-connected, and $G_2 = W_{\text{no}}[G_1, S]$ for some $w$-sequence of $G_1$. Let $X \in S$; we may assume (by possibly swapping $X$ with its complement) that $X$ is simple. Denote by $u$ the vertex in $\mathcal{S}_G(X) = \mathcal{R}_{G_1}(X) = \mathcal{S}_{G_2}(X)$. By Remark 27 we may assume that $X$ is the first element in the sequence $S$. Thus we may assume that $X$ does not preserve $P_1$ for otherwise we may replace $G_1$ with $W_{\text{no}}[G_1, X]$. This implies that $u$ is the endpoint of $P_1$ in $G_1$, hence in $G$. Similarly, applying the argument for $G_2$, we deduce that $u$ is the endpoint of $P_2$ in $G$. Since $T = V_{\text{odd}}(G[P_1 \triangle P_2])$, $u \notin T$, a contradiction as $X$ is simple. Hence $S = \emptyset$, $P_1$ and $P_2$ are paths of $G$, and $|T| \leq 4$.

We will postpone the proof of the following key lemma until the next section.

**Lemma 37.** Every nice graft has a minor that is a nearly 3-connected nice graft.

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Proof of Lemma 15. Let $N, M, F$ and $F'$ be as in the statement of the lemma. We may assume that $F'$ is not an equivalence class. By Lemma 23, $F'$ is the union of two equivalence classes $F_1$ and $F_2$ and any $(G_1, T_1) \in F_1$ and $(G_2, T_2) \in F_2$ are clip siblings. By Theorem 26, $(G_1, T_1)$ and $(G_2, T_2)$ are either basic or nested siblings, however the former is not possible because of Remark 28. Since $F_1$ and $F_2$ are equivalence classes, we can choose $(G_1, T_1) \in F_1$ and $(G_2, T_2) \in F_2$ so that they are nested twins arising from a template $\mathcal{T} = (H_1, P_1, H_2, P_2, S)$.

Let $i \in [2]$. We need to show that $(G_i, T_i)$ has no reaching pair. By Lemma 34, $(G_i, T_i)$ has a minor $(\hat{G}, \hat{T})$ that is nice. By Lemma 37, $(\hat{G}, \hat{T})$ has a minor $(\hat{G}', \hat{T}')$ that is nice and nearly 3-connected. In particular, since it is nice, it is non-degenerate. It follows by Lemma 36 that $(\hat{G}', \hat{T}')$ has no reaching pair. Since $(\hat{G}', \hat{T}')$ is a minor of $(G_i, T_i)$, Remark 13 implies that $(G_i, T_i)$ has no reaching pair. □

5.2 A few observations about 2-separations

For a graph $H$, we say that a sequence $F = (B_1, \ldots, B_t)$ with $t \geq 2$, where $B_1, \ldots, B_t$ is a partition of $E(H)$, is a flower if there exist distinct $u_1, \ldots, u_t \in V(H)$ such that,

- $H[B_i]$ is connected, for every $i \in [t]$, and
- $\mathcal{B}_H(B_i) = \{u_i, u_{i+1}\}$, for every $i \in [t]$ (where $t + 1$ denotes 1).

For $i \in [t]$, $B_i$ is a petal with attachments $u_i$ and $u_{i+1}$. We say that the flower is maximal if for no petal $B$, the graph $H[B]$ has a cut-vertex separating its attachments. For all $i \in [t]$, petals $B_i$ and $B_{i+1}$ are consecutive. Maximal flowers correspond to generalized circuits as introduced by Tutte in [13]. The term flower was introduced to describe crossing 3-separations in matroids (see [6]).

Lemma 38. Let $(H, T)$ be a graft and $F = (B_1, \ldots, B_t)$ be a flower of $H$ with attachments $U$. Suppose that $T = T_a \cup T_b$ where $T_a \subseteq U$, $T_b \cap U = \emptyset$, $|T_b| \leq 4$, and no two vertices of $T_b$ are contained in the same petal of $F$. Then $(H, T)$ is degenerate.

Proof. $|T| = 2k$ for some integer $k$ and we may choose a $T$-join $J = P_1 \triangle \ldots \triangle P_k$ where $P_1, \ldots, P_k$ are pairwise vertex-disjoint paths of $H$. Let $\mathcal{B} = \{B \in F : B \cap P_i \neq \emptyset, \text{for some } i \in [k]\}$. Let $H'$ be obtained from $H$ by rearranging the petals of $F$ so that the petals in $\mathcal{B}$ are consecutive in $H'$. After possible Whitney flips on some of the petals in $H'$ we may obtain a graph $H''$ where $J$ is the union of at most two paths. Let $T'' := V_{odd}(H''[J])$. Then $|T''| \leq 4$ and $(H'', T'')$ is equivalent to $(H, T)$. □

We leave the following observation as an exercise,
Remark 39. Let $H$ be a 2-connected graph and let $X$ and $Y$ be crossing 2-separations of $H$. Then there exists a partition $Z_1, Z_2, Z_3, Z_4$ of the edges of $G$ such that $X = Z_1 \cup Z_2$, $Y = Z_2 \cup Z_3$ and either,

(1) $(Z_1, Z_2, Z_3, Z_4)$ is a flower of $H$, or

(2) $\mathcal{B}_G(Z_4) = \mathcal{B}_G(X) = \mathcal{B}_G(Y)$ for all $i \in [4]$.

Let $(G, T)$ be a graft where $G$ is 2-connected. Let $X$ be a 2-separation of $G$ and denote by $u_1, u_2$ the vertices in $\mathcal{B}_G(X)$. We say that $X$ is a Type I separation if $\mathcal{S}_G(X) \cap T = \emptyset$, i.e. if $X$ is a 2-$(0, i)$ separation for some $i \in \{0, 1\}$. Suppose now that there exists a unique vertex $v$ in $\mathcal{S}_G(X) \cap T$. We say that $X$ is a Type II separation if $v$ is a cut-vertex of $G[X]$ separating $u_1$ and $u_2$. We say that $X$ is a Type III separation if $v$ is not such a cut-vertex, i.e. if there exists a $(u_1, u_2)$-path of $G[X]$ avoiding $v$.

Consider a graft $(G, T)$ with a 2-separation $X$ with $\mathcal{B}_G(X) = \{u_1, u_2\}$. Suppose that $X$ is a Type I separation. Let $(H, T)$ be obtained from $(G, T)$ by replacing $X$ by an edge $u_1 u_2$. We say that $(H, T)$ is obtained from $(G, T)$ by a Type I simplification. Suppose that $X$ is a Type 2 separation where $v$ is the vertex in $\mathcal{S}_G(X) \cap T$. Let $(H, T)$ be obtained from $(G, T)$ by replacing $X$ by edges $u_1 v$ and $v u_2$. We say that $(H, T)$ is obtained from $(G, T)$ by a Type II simplification. Suppose that $X$ is a Type 3 separation where $v$ is the vertex in $\mathcal{S}_G(X) \cap T$. Let $(H, T)$ be obtained from $(G, T)$ by replacing $X$ by edges $u_1 v, v u_2$ and $u_1 u_2$. We say that $(H, T)$ is obtained from $(G, T)$ by a Type III simplification. In all three cases we say that $(H, T)$ is obtained by simplifying separation $X$ of $(G, T)$.

Lemma 40. Let $(G, T)$ be a graft where $G$ is 2-connected, and let $(H, T)$ be obtained from $(G, T)$ by simplifying a separation. Then

(1) $(H, T)$ is a minor of $(G, T)$ and

(2) if $(G, T)$ is non-degenerate then $(H, T)$ is non-degenerate.

Proof. (1) Let $X$ be a 2-separation of $G$ with $\mathcal{B}_G(X) = \{u_1, u_2\}$. Suppose that $X$ is a Type I separation. Since $G$ is 2-connected, there exists a $(u_1, u_2)$-path $P$ in $G[X]$. Then we obtain a Type I simplification by contracting all but one edge of $P$. Suppose that $X$ is a Type II separation. Since $G$ is 2-connected, there exists a $(u_1, u_2)$-path $P$ in $G[X]$ using vertex $v$. Then we obtain a Type II simplification by contracting all edges of $P$ that are not adjacent to $v$. Suppose that $X$ is a Type III separation. Since $G$ is 2-connected, there exists a $(u_1, u_2)$ path $P$ in $G[X]$ using vertex $v$. Since $v$ is not a cut-vertex of $G[X]$ separating $u_1$ and $u_2$, there exists a $(u_1, u_2)$-path $Q$ in $G[X]$ avoiding $v$. For $i = 1, 2$ let $z_i$ be the last vertex of $Q$ in the subpath $P(u_i, v)$ starting from $u_i$. Then we obtain a Type III simplification by deleting all edges

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4 If $P$ is a path with vertices $a, b$, then $P(a, b)$ denotes the subpath of $P$ between vertices $a$ and $b$. 27
of \( X \) outside \( P \cup Q(z_1,z_2) \) and by contracting all edges in \( P(u_1,z_1) \cup P(u_2,z_2) \), all edges in \( P(z_1,z_2) \) not incident to \( v \) and all but one edge of \( Q(z_1,z_2) \). (2) follows from the fact that \((H,T)\) and \((G,T)\) have the same set of terminals and every 2-separation of \( H \) corresponds to a 2-separation of \( G \). Hence, if there exists a sequence of Whitney-flips that transforms the graft \((H,T)\) into a graft with at most four terminals, then the corresponding Whitney-flips on \((G,T)\) would transform \((G,T)\) into a graft with at most four terminals.

5.3 The proof of Lemma 37

We are now ready to prove the last lemma of the paper.

**Proof of Lemma 37.** Let \((G,T)\) be a nice graft and let \( \Omega, H = G \setminus \Omega, S = (X_1, \ldots, X_k), v_1, v_2, w_1, w_2, T \) be as in the definition of nice grafts (see section 5.1). We may assume (by choosing a minor minimal example) that no minor of \((G,T)\) is equivalent to a nice graft. We will use throughout this proof the fact that by (B5) \( H \) is 2-connected and that since \( \Omega \) is not a bridge, so is \( G \).

We need to show that \((G,T)\) is nearly 3-connected. Suppose for a contradiction this is not the case i.e. there exists a 2-separation \( Y \) of \((G,T)\) with \( \Omega \notin Y \) where neither \( Y \), nor \( \bar{Y} \) is simple. We will show that for some \((G',T')\) equivalent to \((G,T)\) and for some \( Y' \subseteq Y \) that is a non-simple 2-separation of \( G' \), the following properties hold:

- (P1) \( Y' \) is a non-simple separation of \((G',T')\) of Type I, II, or III; and
- (P2) the graft obtained from \((G',T')\) by simplifying the separation \( Y' \) is equivalent to a nice graft.

Then (P1) and (P2) will contradict the fact that no minor of \((G,T)\) is equivalent to a nice graft.

Note that we may swap the role of \( X_1 \) and \( \bar{X}_k \), as we may replace \( S \) by \((\bar{X}_k, \bar{X}_{k-1}, \ldots, \bar{X}_1) \). Throughout the proof we will make repeated use of this symmetry.

**Claim 1.** We may assume that \( \mathcal{J}_G(Y) \cap T \neq \emptyset \).

**Proof.** For otherwise \( Y \) is a Type I separation and let \((G',T')\) be obtained from \((G,T)\) by simplifying \( Y \). Lemma 40 implies that \((G',T)\) is a minor of \((G,T)\) and that \((G',T)\) is non-degenerate. It is easy to verify that properties (B1)-(B5) are preserved for \((G',T')\). Hence, (P1) and (P2) hold as required. \( \diamond \)

By hypothesis, \( \Omega = v_1w_1 \notin Y \). By (B1), \( v_1 \in \mathcal{J}_H(X_1) \), and \( w_1 \in \mathcal{J}_H(\bar{X}_k) \). Thus, \( X_1 - Y \neq \emptyset \) and \( \bar{X}_k - Y \neq \emptyset \). In particular one of the following three cases must hold.

**Case 1:** \( Y \subseteq X_1 \) or \( Y \subseteq \bar{X}_k \).
Since we can interchange the role of $X_1$ and $\tilde{X}_k$, it suffices to consider the case where $Y \subseteq X_1$. Then $v_1 \notin \mathcal{I}_H(Y)$ as the edge $\Omega$ is incident to $v_1$. By Claim 1, $T \cap \mathcal{I}_H(Y)$ is non-empty. (B1) and (B2) imply that $T \cap \mathcal{I}_H(X_1) = \{v_1, v_2\}$. It follows that $T \cap \mathcal{I}_H(Y) = \{v_2\}$. If $v_2$ is a cut-vertex of $G[Y]$ separating $\mathcal{B}_G(Y)$ then $Y$ is a separation of Type II, otherwise $Y$ is a separation of Type III. Hence, we proved (P1). Simplify $Y$ by a simple separation $Y'$, and let $X'_1 = (X_1 - Y) \cup Y'$. Using Lemma 40 it is easy to verify that properties (B1)-(B5) are preserved, using $X_1'$ instead of $X_1$. Thus (P2) holds as required.

**Case 2:** $Y \subseteq X_k - X_1$.

By Claim 1, there exists $x \in \mathcal{I}_H(Y) \cap T$. By (B2), $x \in \mathcal{B}_H(X_p)$ for some $p \in [k]$. Note that $p \neq 1, k$, as $x \in \mathcal{I}_H(Y)$ and $Y \subseteq X_k - X_1$. As $x \in \mathcal{I}_H(Y)$, the sets $X_p \cap Y$ and $Y - X_p$ are non-empty. Moreover, $X_1 \cap X_p$ and $Y \cap X_1$ is empty, so $X_p - Y$ is also non-empty. Finally $\tilde{X}_k$ is contained in both $\tilde{X}_p$ and in $Y$, hence $\tilde{X}_p \cap Y$ is non-empty. It follows that $Y$ and $X_p$ cross. Therefore there exists a partition $Z_1, Z_2, Z_3, Z_4$ of $E(H)$ such that $X_p = Z_1 \cup Z_2$, $Y = Z_2 \cup Z_3$ and by Remark 39 one of the following occurs:

$$(\alpha) \quad \mathcal{B}_H(Z_i) = \mathcal{B}_H(X_p) = \mathcal{B}_H(Y), \text{ for every } i \in [4],$$

$$(\beta) \quad (Z_1, Z_2, Z_3, Z_4) \text{ is a flower of } H.$$  

However, (\alpha) does not occur, as $x \in \mathcal{B}_H(X_p) - \mathcal{B}_H(Y)$; therefore (\beta) occurs. Since $Y = Z_2 \cup Z_3$, $Z_2$ and $Z_3$ are consecutive petals with common attachment $x$. In particular, we have shown that every vertex in $\mathcal{I}_H(Y) \cap T$ is a cut vertex of $H[Y]$ separating the vertices in $\mathcal{B}_H(Y)$. Applying this last result to each vertex in $\mathcal{I}_H(Y) \cap T$ it follows that $Y$ can be partitioned into sets $B_1, \ldots, B_\ell$, for some $\ell \geq 2$, such that $(Z_1, B_1, \ldots, B_\ell, Z_4)$ form a flower. Moreover, the attachments of consecutive petals of $B_1, \ldots, B_\ell$ are in $T$ and, for all $i \in [\ell]$, $\mathcal{B}_H(B_i) \cap T = \emptyset$. It follows by Claim 1 (applied to each $B_i$) that, for all $i \in [\ell]$, $B_i$ consists of a single edge, say $e_i$, and that $e_1, \ldots, e_\ell$ are series edges. Since $Y$ is not simple, $\ell \geq 3$. Let $(G', T')$ be obtained from $(G, T)$ by a Whitney flip on $\{e_1, e_2\}$. In $(G', T')$ (see Remark 30) $e_2$ and $e_3$ are series edges and the vertex common to these edges in not in $T'$. Hence, $Y' = \{e_2, e_3\}$ is a non-simple separation of Type I of $(G', T')$ and (P1) is satisfied. Proceeding as in Claim 1 we prove that (P2) holds as well.

**Case 3:** $Y$ crosses $X_1$ or $Y$ crosses $\tilde{X}_k$.

Since we can interchange the role of $X_1$ and $\tilde{X}_k$, it suffices to consider the case where $Y$ crosses $X_1$. Therefore there exists a partition $Z_1, Z_2, Z_3, Z_4$ of $E(H)$ such that $X_1 = Z_1 \cup Z_2$, $Y = Z_2 \cup Z_3$ and, by Remark 39, one of the following occurs:
(α) \( \mathcal{R}_H(Z_i) = \mathcal{R}_H(X_1) \), for every \( i \in [4] \), or

(β) \((Z_1, Z_2, Z_3, Z_4)\) is a flower of \( H \).

As \( X_1 = Z_1 \cup Z_2 \) and \( \Omega = v_1 w_1 \notin Y \), we have \( v_1 \in V_H(Z_1) \) and \( w_1 \in V_H(Z_4) \).

**Claim 2.** Case (α) does not occur.

**Proof.** Suppose for a contradiction that case (α) occurs. We first claim that \( \mathcal{I}_H(Z_3) \cap T \neq \emptyset \). Suppose that this is not the case; then, because of Claim 1, \( Z_3 \) consists of a single edge. As we obtained a contradiction in Case 1 for non-simple 2-separations, \( Z_2 \) is a simple separation or consists of a single edge. By Claim 1, \( H \) does not have parallel edges. Thus \( Y = Z_2 \cup Z_3 \) is simple, a contradiction.

Suppose that \( k \geq 2 \), i.e. there is a 2-separation \( X_2 \in \mathcal{S} \) with \( X_1 \subset X_2 \). We may assume that \( X_1 \) and \( X_2 \) have distinct boundaries. Let \( \{a_1, a_2\} \) denote the vertices in \( \mathcal{R}_H(X_2) \). As \( X_1 \subset X_2 \), \( a_1, a_2 \in V_H(Z_3 \cup Z_4) \). If \( a_1, a_2 \in V_H(Z_4) \), then \( Z_3 \subset X_2 - X_1 \) and \( T \cap \mathcal{I}_H(Z_3) \) is empty, a contradiction. Hence, we may assume \( a_1 \in \mathcal{I}_H(Z_3) \); then (as we are in case (α) and since \( H \) is 2-connected) there exists a \((v_1, w_1)\)-path in \( H \setminus \{a_1, a_2\} \), a contradiction. Thus \( k = 1 \), i.e. \( \mathcal{S} = (X_1) \); hence, by (B2), \( T \subseteq \{v_1, v_2, w_1, w_2\} \cup \mathcal{R}_H(X_1) \). As \( |T| \geq 6 \) (by (B4)), we have \( T = \{v_1, v_2, w_1, w_2\} \cup \mathcal{R}_H(X_1) \). By Remark 30, it follows that \( W_{w_0}(H, T, X_1) \) has four terminals, contradicting (B4).

Thus we may assume case (β) occurs.

**Claim 3.** \( Y \) does not cross \( \hat{X}_k \).

**Proof.** Suppose for a contradiction it does. Then, by a similar argument to the one above (applied to \( \hat{X}_k \)), there exists a flower \((W_1, W_2, W_3, W_4)\) of \( H \) with \( \hat{X}_k = W_1 \cup W_2 \) and \( Y = W_3 \cup W_4 \). Let \( F \) be the maximal flower that is obtained from the flower \((Z_1, Z_2, Z_3, Z_4)\) by partitioning the petals \( Z_i \) into as many petals as possible and let \( F' \) be the maximal flower that is a obtained from the flower \((W_1, W_2, W_3, W_4)\) by partitioning the petals \( W_i \) into as many petals as possible. As \( Y \) crosses both \( X_1 \) and \( \hat{X}_k \), we have \( F = F' \). Hence \( F \) is a flower of \( H \) and \( X_1 \) and \( \hat{X}_k \) are each the union of at least two petals of \( F \). We may assume that \( H[X_1] \) does not partition into sets \( U_1, U_2 \), where \( v_1, v_2 \in \mathcal{I}_H(U_1) \) and \( (U_1, U_2, \hat{X}_1) \) is a flower of \( H \), as otherwise we may redefine \( X_1 \) to be \( U_1 \) (by adding \( U_1 \) to the sequence \( \mathcal{S} \)). The analogue statement holds for \( \hat{X}_k \). It follows that \( v_1 \) and \( v_2 \) are not in the interior of the same petal of \( F \). Similarly, \( w_1 \) and \( w_2 \) are not in the interior of the same petal of \( F \). Moreover, as we obtained \( F \) from \((Z_1, Z_2, Z_3, Z_4)\), \( v_1 \) and \( w_1 \) are in distinct petals of \( F \). For every \( X \in \mathcal{S} \), the vertices in \( \mathcal{R}_H(X) \) are attachments of \( F \), as there is no \((v_1, w_1)\)-path in \( H - \mathcal{R}_H(X) \). Hence \( T_e \) is contained in the set of attachments of \( F \). By Lemma 38, \((H, T)\) is degenerate, a contradiction to (B4).
Thus $Y$ does not cross $\bar{X}_k$. Hence, $Y \cap \bar{X}_k = \emptyset$ and thus $Z_3 \subseteq X_k - X_1$. As we obtained a contradiction in Case 2 for non-simple 2-separations, $Z_3$ either consists of a single edge $e$, or $Z_3$ is simple. If $Z_3$ is simple, the vertex in $T \cap \mathcal{I}_H(Z_3)$ is a cut-vertex of $H[Z_3]$, because of condition (B2). Therefore either $Z_3$ consists of a single edge $e$ or it consists of two series edges, say $e, f$ incident to a vertex $x \in T$. If $Z_2$ consists of a single edge, say $g$, then, as $Y$ is non-simple, $Z_3 = \{e, f\}$. It follows that $e, f, g$ are three series edges and we obtain a contradiction as in the proof of Case 2.

Thus we may assume that $|Z_2| > 1$. Because of Claim 1, $\mathcal{I}_H(Z_2) \neq \emptyset$. As we obtained a contradiction in Case 1 for non-simple 2-separations, $Z_2$ is simple, hence it is a Type II or Type III separation. We may assume that $e$ has an endpoint $y$ in $\mathcal{B}_G(Z_2)$.

If $y \notin T$, then, by Lemma 40, properties (P1) and (P2) hold as required. Hence we may assume that $y \in T$. Let $(G', T') = \text{Wflip}[(G, T), Z_2]$. Denote by $y'$ the endpoint of $e$ that is in $\mathcal{B}_G(Z_2)$ in $G'$. Since $Z_2$ is a Type II or Type III separation, $|\mathcal{I}_G(Z_2) \cap T| = 1$. Thus, by Remark 30, $y' \notin T'$. It follows that $Z_2 \cup \{e\}$ is a Type II or Type III separation of $(G', T')$. Thus property (P1) holds. A Type II or Type III simplification will replace $(G', T')$ by $(H, S) = (G', T')/e$. Observe, that $\text{Wflip}[(H, S), Z_2] = (G, T)/e$. Using Lemma 40 it follows readily that $(G, T)/e$ is nice, hence property (P2) holds as required.

References


