Abstract

A signed graph is a representation of an even cycle matroid $M$ if the cycles of $M$ correspond to the even cycles of that signed graph. Two, long standing, open questions regarding even cycle matroids are the problem finding an excluded minor characterization and the problem of efficiently recognizing this class of matroids. Progress on these problems has been hampered by the fact that even cycle matroids can have an arbitrary number of pairwise inequivalent representations (two signed graph are equivalent if they are related by a sequence of Whitney-flips and signature exchanges). We show that we can bound the number of inequivalent representations of an even cycle matroid $M$ (under some mild connectivity assumptions) if $M$ contains any fixed size minor that is not a projection of a graphic matroid. For instance, any connected even cycle matroid which contains $R_{10}$ as a minor has at most 6 inequivalent representations.

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1 Introduction

We assume that the reader is familiar with the basics of matroid theory. See Oxley [7] for the definition of the terms used here. We will only consider binary matroids in this paper. Thus the reader should substitute the term “binary matroid” every time “matroid” appears in this text.

In this article, we will consider graphs with multiple edges and loops. Let $G$ be a graph. For a set $X \subseteq E(G)$, we write $V_G(X)$ to refer to the set of vertices incident to an edge of $X$ and $G[X]$ for the subgraph with vertex set $V_G(X)$ and edge set $X$. A subset $C$ of edges is a cycle if $G[C]$ is a graph where every vertex has even degree. An inclusion-wise minimal non-empty cycle is a circuit. Let $G$ be a graph. We denote by cycle$(G)$ the set of all cycles of $G$. The set cycle$(G)$ is the set of cycles of the graphic matroid of $G$. We identify cycle$(G)$ with that matroid. We say that $G$ is a representation of that graphic matroid.

A signed graph is a pair $(G, \Sigma)$ where $G$ is a graph and $\Sigma \subseteq E(G)$. A subset $B \subseteq E(G)$ is even (resp. odd) if $|B \cap \Sigma|$ is even (resp. odd). In particular an edge $e$ is even if and only if $e \in \Sigma$. We say that $\Sigma'$ is a signature of $(G, \Sigma)$ if $(G, \Sigma)$ and $(G, \Sigma')$ have the same set of even cycles. Equivalently, it is straightforward to prove that $\Sigma'$ is a signature if $\Sigma \Delta \Sigma'$ is a cut of $G$. In that case $(G, \Sigma)$ and $(G, \Sigma')$ are related by a signature exchange. Let $(G, \Sigma)$ be a signed graph. We denote by ecycle$(G, \Sigma)$ the set of all even cycles of $(G, \Sigma)$. The set ecycle$(G, \Sigma)$ is the set of cycles of a binary matroid known as the even-cycle matroid. We identify ecycle$(G, \Sigma)$ with that matroid. We say that $(G, \Sigma)$ is a representation of that matroid. Observe that since cycle$(G) = \text{ecycle}(G, \emptyset)$, every graphic matroid is an even cycle matroid.

1.1 Representations of graphic matroids are nice

We will state a theorem that shows, for a graphic matroid, how to construct the set of all representations from a single representation. We require a number of definitions.

Let $G$ be a graph and let $X \subseteq E(G)$. We write $\mathcal{R}_G(X)$ for $V_G(X) \cap V_G(\bar{X})$. Suppose that $\mathcal{R}_G(X) = \{u_1, u_2\}$ for some $u_1, u_2 \in V(G)$. Let $G'$ be the graph obtained by identifying vertices $u_1, u_2$ of $G[X]$ with vertices $u_2, u_1$ of $G[\bar{X}]$ respectively. Then $G'$ is obtained from $G$ by a Whitney-flip on $X$. We will also call Whitney-flip the operation consisting of identifying two vertices from distinct components, or the operation consisting of partitioning the graph into components each of which is a block of $G$. We define $\bar{X} = E(G) - X$, where for any pair of sets $A$ and $B$, $A - B = \{a \in A : a \notin B\}$. Throughout the paper we shall omit indices when there is no ambiguity. For instance we may write $\mathcal{R}(X)$ for $\mathcal{R}_G(X)$. 

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two graphs to be *equivalent* if one can be obtained from the other by a sequence of Whitney-flips (it is easy to verify that this does indeed define an equivalence relation).

In a seminal paper [14], Whitney proved the following.

**Theorem 1.** A graphic matroid has a unique representation, up to equivalence.

It follows in particular that, if a graphic matroid is 3-connected, then it has a unique representation.

### 1.2 Representations of even cycle matroids are naughty

The situation is considerably more complicated for even cycle matroids than for graphic matroids as we will illustrate in this section.

Suppose that \((G_1, \Sigma_1)\) and \((G_2, \Sigma_2)\) are signed graphs where \(G_1\) and \(G_2\) are equivalent and \(\Sigma_2\) is obtained from \(\Sigma_1\) by a signature exchange. Then we say that \((G_1, \Sigma_1)\) and \((G_2, \Sigma_2)\) are *equivalent*. It can be easily checked that if \(G_1\) and \(G_2\) are equivalent graphs and \(\text{ecycle}(G_1, \Sigma_1) = \text{ecycle}(G_2, \Sigma_2)\) for some signatures \(\Sigma_1\) and \(\Sigma_2\), then \((G_1, \Sigma_1)\) and \((G_2, \Sigma_2)\) are equivalent. Equivalent signed graphs do indeed define an equivalence relation. It follows that for any even cycle matroid \(N\) we can partition its representations into equivalence classes \(F_1, \ldots, F_k\). We will say that \(F_i\) \((i \in [k])\) is an *equivalence class of \(N\)*.

There is no direct analogue to Whitney’s theorem for even cycle matroids as the following result illustrates,

**Remark 2.** For any integer \(k\), there is a even cycle matroid \(M\) with \(|E(M)| \leq 5k\) and \(2^k\) equivalence classes.

We now describe a general operation to construct the matroids given in the previous result.

Let \(G\) be a graph. Given \(U \subseteq V(G)\), we denote by \(\delta_G(U)\) the *cut* induced by \(U\), that is \(\delta_G(U) := \{(u, v) \in E(G): u \in U, v \notin U\}\). We write \(\delta_G(u)\) for \(\delta_G(\{u\})\). Given a graph \(G\) we denote by \(\text{loop}(G)\) the set of all loops of \(G\). Let \((G, \Sigma)\) be a signed graph. A vertex \(s\) is a *blocking vertex* of \((G, \Sigma)\) if every odd circuit of \((G, \Sigma)\) either contains the vertex \(s\) or is a loop. Similarly, a pair of vertices \(s, t\) is a *blocking pair* if every odd circuit of \((G, \Sigma)\) either uses at least one of \(s\) and \(t\) or is a loop. Note that \(s\) is a blocking vertex (respectively \(s, t\) is a blocking pair) of \((G, \Sigma)\) if and only if there exists a signature \(\Sigma'\) of \((G, \Sigma)\) such that \(\Sigma' \subseteq \delta(s) \cup \text{loop}(G)\) (respectively \(\Sigma' \subseteq \delta(s) \cup \delta(t) \cup \text{loop}(G)\)).
Consider a signed graph \((G, \Sigma)\) and vertices \(v_1, v_2 \in V(G)\), where \(\Sigma \subseteq \delta_G(v_1) \cup \delta_G(v_2) \cup \text{loop}(G)\). So \(v_1, v_2\) is a blocking pair of \((G, \Sigma)\). We can construct a signed graph \((G', \Sigma)\) from \((G, \Sigma)\) by replacing the endpoints \(x, y\) of every odd edge \(e\) with new endpoints \(x', y'\) as follows:

- if \(x = v_1\) and \(y = v_2\) then \(x' = y'\) (i.e. \(e\) becomes a loop);
- if \(x = y\) (i.e. \(e\) is a loop), then \(x' = v_1\) and \(y' = v_2\);
- if \(x = v_1\) and \(y \neq v_1, v_2\), then \(x' = v_2\) and \(y' = y\);
- if \(x = v_2\) and \(y \neq v_1, v_2\), then \(x' = v_1\) and \(y' = y\).

Then we say that \((G', \Sigma)\) is obtained from \((G, \Sigma)\) by a Lovász-flip on \(v_1, v_2\). It is easy to show that Lovász-flips preserve even cycles \([3, 4]\). Using Lovász-flips we can construct inequivalent signed graphs representing the same even cycle matroid. An example is given in Figure 1. Each \(G_1, \ldots, G_4\) may stand for an arbitrary graph. As an example we chose \(G_1\) to be the graph with edges 1, 2, 3, 4, 5, 6 given in the figure. The arrows indicate how each piece is flipped between the graph on the left and the graph on the right. The odd edges, in both signed graphs, are 1, 2, 3. Note that, for every \(i \in [4]\), the two vertices in \(V_{G_i} \cap V_{G_{i+1}}\) form a blocking pair. It is possible to obtain the signed graph on the right from the signed graph on the left by signature exchanges and Lovász-flips on each of these blocking pairs. This construction generalizes to any number of graphs \(G_1, \ldots, G_k\) and using Lovász-flips and signature exchanges we can flip any subset of these \(k\) graphs. In particular, it is easy to construct matroids \(M\) as in Remark 2.

![Figure 1: Inequivalent signed graphs representing the same matroid.](image-url)
As Remark 2 shows, if a signed graph \((G, \Sigma)\) has blocking pairs then \(\text{ecycle}(G, \Sigma)\) may have many inequivalent representations. On the other hand, if a signed graph has a blocking pair, then it cannot have three, pairwise vertex disjoint, odd circuits. Thus one may wonder if having three, pairwise vertex disjoint, odd circuits, forces the representation to be unique, up to equivalence. Slilaty [11] proved that the analogue of this statement holds for \textit{signed-graphic} matroids. Alas, no similar result holds for even cycle matroids, as the following remark indicates. It shows that blocking pairs are not the only reason for having inequivalent representations.

**Remark 3.** For every integer \(k\), there exists a signed graph \((G, \Sigma)\) with the property that:

1. every signed graph equivalent to \((G, \Sigma)\) has \(k\), pairwise vertex disjoint, odd circuits, and
2. \(\text{ecycle}(G, \Sigma)\) has at least two inequivalent representations.

We postpone the proof of this remark until Section 4.4.

### 1.3 Main results

Given a matroid \(M\) and disjoint subsets \(I, J \subseteq E(M)\), the matroid \(M \setminus I / J\) denotes the minor of \(M\) obtained by deleting the elements in \(I\) and contracting the elements in \(J\). We define minor operations on signed graphs as follows. Let \((G, \Sigma)\) be a signed graph and let \(e \in E(G)\). Then \((G, \Sigma) \setminus e\) is defined as \(\langle G \setminus e, \Sigma - \{e\} \rangle\). \(^2\) We define \((G, \Sigma) / e\) as \((G \setminus e, \emptyset)\) if \(e\) is an odd loop of \((G, \Sigma)\) and as \((G \setminus e, \Sigma)\) if \(e\) is an even loop of \((G, \Sigma)\); otherwise \((G, \Sigma) / e\) is equal to \((G/e, \Sigma')\), where \(\Sigma'\) is any signature of \((G, \Sigma)\) which does not contain \(e\). Observe that (see [6] for instance),

**Remark 4.** \(\text{ecycle}(G, \Sigma) \setminus I / J = \text{ecycle}(\langle G, \Sigma \rangle \setminus I / J)\).

In particular, this implies that being an even cycle matroid is a minor closed property.

#### 1.3.1 Non-degenerate minors

We say that an even cycle matroid is \textit{degenerate} if any of its representation has a blocking pair. If a signed graph has a blocking pair, then so does every minor. It follows from Remark 4 that being degenerate is a minor closed property. If an even cycle matroid \(N\) is graphic, then it has a representation \((G, \emptyset)\) as an

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\(^2\)Given a graph \(H\) and \(e \in E(H)\), \(G \setminus e\) is the graph obtained by deleting \(e\) whereas \(G/e\) is the graph obtained by contracting \(e\).
even cycle matroid, and trivially, any pair of vertices of \( G \) is a blocking pair. Hence, graphic matroids are degenerate. An example of an even cycle matroid which is non-degenerate is given by the matroid \( R_{10} \) (introduced in [10]). \( R_{10} \) has six representations as an even cycle matroid, all isomorphic to the signed graph \( (K_5, E(K_5)) \). (How to find these representations is explained in [6].) This signed graph does not have a blocking pair, as the removal of any two vertices leaves an odd triangle.

We are now ready to present the first main result of the paper,

**Theorem 5.** Let \( M \) be a 3-connected even cycle matroid which contains as a minor a non-degenerate 3-connected matroid \( N \). Then the number of equivalence classes of \( M \) is at most twice the number of equivalence classes of \( N \).

This result implies, in particular, that every 3-connected even cycle matroid containing \( R_{10} \) as a minor has, up to equivalence, at most 12 representations. We will strengthen this result in Section 1.3.2.

We will show that degenerate matroids are “close” to being graphic matroids. We require a number of definitions to formalize this notion.

Consider a graph \( H \) with a vertex \( v \) and \( \alpha \subseteq \delta_H(v) \cup \text{loop}(H) \). We say that \( G \) is obtained from \( H \) by splitting \( v \) into \( v_1 \) and \( v_2 \) according to \( \alpha \) if \( V(G) = V(H) - \{v\} \cup \{v_1, v_2\} \) and for every \( e = (u, w) \in E(H) \):

- if \( e \not\in \alpha \cup \delta_H(v) \), then \( e = (u, w) \) in \( G \);
- if \( e \in \text{loop}(H) \cap \alpha \), then \( e = (v_1, v_2) \) in \( G \);
- if \( e \in \delta_H(v) \cap \alpha \) and \( u \neq v, w = v \), then \( e = (u, v_1) \) in \( G \);
- if \( e \in \delta_H(v) - \alpha \) and \( u \neq v, w = v \), then \( e = (u, v_2) \) in \( G \).

Let \( N \) and \( M \) be matroids where \( E(N) = E(M) \). Then \( N \) is a lift of \( M \) if, for some matroid \( M' \) where \( E(M') = E(M) \cup \{\Omega\} \), \( M = M' / \Omega \) and \( N = M' \setminus \Omega \). If \( N \) is a lift of \( M \) then \( M \) is a projection of \( N \). Lifts and projections were introduced in [2]. Every even cycle matroid \( M \) is a lift of a graphic matroid; indeed, for any representation \((G, \Sigma)\) of \( M \) we may construct \((G', \Sigma')\) by adding an odd loop \( \Omega \). Then \( \text{ecycle}(G', \Sigma') / \Omega \) is a graphic matroid. The following result shows that degenerate even cycle matroids are projections of graphic matroids.

**Remark 6.** Let \((H, \Gamma)\) be a signed graph.
(1) If \((H, \Gamma)\) has a blocking vertex, then \(\text{ecycle}(H, \Gamma)\) is a graphic matroid.

(2) If \((H, \Gamma)\) has a blocking pair, then \(\text{ecycle}(H, \Gamma)\) is a projection of a graphic matroid.

Proof. (1) Suppose that \(\Gamma \subseteq \delta_H(s) \cup \text{loop}(H)\) for some vertex \(s\) of \(H\). Let \(G\) be obtained from \(H\) by splitting \(s\) according to \(\Gamma\). Then \(\text{cycle}(G) = \text{ecycle}(H, \Gamma)\). (2) Suppose that \(\Gamma \subseteq \delta_H(s) \cup \delta_H(t) \cup \text{loop}(H)\) for a pair of vertices \(s, t\) of \(H\). Let \(G\) be obtained from \(H\) by splitting \(s\) into \(s_1, s_2\) according to \(\delta_H(s) \cap \Gamma\) and by adding an edge \(\Omega = (s_1, s_2)\). Let \(M' = \text{ecycle}(G, \Gamma)\). Then by construction \((G, \Gamma)/\Omega = (H, \Gamma)\), hence \(M'/\Omega = M\). Moreover, by (1), \(\text{ecycle}(G, \Gamma) \setminus \Omega = M' \setminus \Omega\) is a graphic matroid, as \(t\) is a blocking vertex of \((G, \Gamma) \setminus \Omega\).

1.3.2 Substantial minors

Consider a signed graph \((G, \Sigma)\) and suppose that there exists a partition \(\mathcal{C}_1, \mathcal{C}_2\) of the odd circuits of \((G, \Sigma)\) and graphs \(G_1\) and \(G_2\) equivalent to \(G\) such that, for \(i = 1, 2\), some \(v_i \in V(G_i)\) intersects all circuits in \(\mathcal{C}_i\). Then we call the pair \((G_1, v_1)\) and \((G_2, v_2)\) an intercepting pair for \((G, \Sigma)\). In particular, if \((G, \Sigma)\) has a blocking pair \(v_1, v_2\), then \((G, v_1)\), \((G, v_2)\) is an intercepting pair for \((G, \Sigma)\). An even cycle matroid is substantial if none of its representations has an intercepting pair. Hence, if an even cycle matroid is degenerate it is not substantial. In particular, substantial matroids are not graphic. We will see (Remark 13) that not being substantial is also a minor closed property. If, for every representation \((G, \Sigma)\) of an even cycle matroid \(M\), the graph \(G\) is 3-connected and \((G, \Sigma)\) has no blocking pair, then \(M\) is substantial. As all 6 representations of \(R_{10}\) are isomorphic to \((K_5, E(K_5))\), \(R_{10}\) is substantial.

We are now ready to present the second main result of the paper,

**Theorem 7.** Let \(M\) be a connected even cycle matroid which contains as a minor a connected matroid \(N\) that is substantial. Then the number of equivalence classes of \(M\) is at most the number of equivalence classes of \(N\).

This result implies, in particular, that every 3-connected even cycle matroid containing \(R_{10}\) as a minor has, up to equivalence, at most 6 representations.

1.4 Related results and motivation

Even-cycle matroids are a natural class of matroids to study as they are the smallest minor closed class of binary matroids which contains all single element co-extensions of graphic matroids. Robertson and
Seymour [8] proved that for every infinite set of graphs one of its members is isomorphic to a minor of another. Gerards, Geelen, and Whittle announced that an analogous result holds for binary matroids. Hence, any minor closed class of binary matroids can be characterized by a finite set of excluded minors. In particular this is the case for even cycle matroids. Tutte [13] gave an explicit description of the excluded minors for the class of graphic matroids. He also gave a polynomial time algorithm to check if a binary matroid (given by its 0,1 matrix representation) is graphic [12].

No explicit description of the excluded minors is known for even cycle matroids and we do not know how to recognize efficiently whether a given binary matroid is an even cycle matroid. The difficulty for both problems lies with the fact that we do not have a sufficient understanding of the representations of even cycle matroids. Theorems 5 and 7 are a first step towards a better understanding of this problem. Eventually, we wish to extend the aforementioned theorems so as to have a compact description of the representations of arbitrary even cycle matroids. We believe that there exists a constant $k$ such that every even cycle matroid with more than $k$ inequivalent representations is constructed in a way analogous to that of the example in Section 1.2. The problem of describing the pairwise relationship between any two representations of an even cycle matroid was addressed in [4] and [5].

1.5 Organization of the paper

Section 2 introduces generalizations of Theorems 5 and 7. An outline of the proofs of these theorems is then presented leaving out two key lemmas, namely 15 and 16. Lemma 15 is proved in Section 3. Section 4 reviews a characterization of class of inequivalent representations of even-cycle matroids. This is required for the proof of Lemma 16 that is given in Section 5.

2 The proofs (modulo the exclusion of several lemmas)

If $N$ is a minor of a matroid $M$ then $M$ is a major of $N$. Consider an even cycle matroid $M$ with a representation $(G, \Sigma)$. Let $I$ and $J$ be disjoint subsets of $E(M)$ and let $N := M \setminus I/J$. Let $(H, \Gamma) := (G, \Sigma) \setminus I/J$. It follows from Remark 4 that $(H, \Gamma)$ is a representation of $N$. We say that $(G, \Sigma)$ is an extension to $M$ of the representation $(H, \Gamma)$ of $N$, or alternatively that $(H, \Gamma)$ extends to $M$.

The following result implies Theorem 5.
Theorem 8. Let $N$ be a 3-connected non-degenerate even cycle matroid. Let $M$ be a 3-connected major of $N$. For every equivalence class $\mathcal{F}$ of $N$, the set of extensions of $\mathcal{F}$ to $M$ is the union of at most two equivalence classes.

The following result implies Theorem 7.

Theorem 9. Let $N$ be a connected substantial even cycle matroid. Let $M$ be a connected major of $N$. For every equivalence class $\mathcal{F}$ of $N$, the set of extensions of $\mathcal{F}$ to $M$ is contained in one equivalence class.

The proofs of Theorems 8 and 9 are constructive. Thus, given a description of the inequivalent representations of $N$, it is possible to construct the set all inequivalent representations of $M$.

2.1 Definitions

First an easy observation,

Remark 10. If $G_1$ and $G_2$ are equivalent graphs, then $G_1 \setminus I/J$ and $G_2 \setminus I/J$ are equivalent.

Proof. Since $G_1$ and $G_2$ are equivalent, $\text{cycle}(G_1) = \text{cycle}(G_2)$. Hence, $\text{cycle}(G_1) \setminus I/J = \text{cycle}(G_2) \setminus I/J$. As the minor operations on graphs and matroid commute, we have, $\text{cycle}(G_1 \setminus I/J) = \text{cycle}(G_2 \setminus I/J)$. The result now follows from Theorem 1.

Consider a matroid $M$ and let $N := M \setminus I/J$ be a minor of $M$. If $I = \emptyset$ and $|I| = 1$ then $M$ is a column major of $N$. If $I = \emptyset$ and $|J| = 1$ then $M$ is a row major of $N$. A set $\mathcal{F}$ of representations of an even cycle matroid is closed under equivalence if, for every $(H, \Gamma) \in \mathcal{F}$ and $(H', \Gamma')$ equivalent to $(H, \Gamma)$, we have that $(H', \Gamma') \in \mathcal{F}$.

Remark 11. Let $\mathcal{F}$ be a set of representations of an even cycle matroid $N$ and let $M$ be a major of $N$. If $\mathcal{F}$ is closed under equivalence, then so is the set $\mathcal{F}'$ of extensions of $\mathcal{F}$ to $M$.

Proof. Let $(G, \Sigma) \in \mathcal{F}'$ and let $(G', \Sigma')$ be equivalent to $(G, \Sigma)$. We have $N = M \setminus I/J$, for some $I, J \subseteq E(M)$. It follows from Remark 10 that $(H, \Gamma) := (G, \Sigma) \setminus I/J$ and $(H', \Gamma') := (G', \Sigma') \setminus I/J$ are equivalent. Since $(G, \Sigma) \in \mathcal{F}'$, Remark 4 implies that $(H, \Gamma) \in \mathcal{F}$. As $\mathcal{F}$ is closed under equivalence, $(H', \Gamma') \in \mathcal{F}$. Hence, by definition, $(G', \Sigma') \in \mathcal{F}'$. 

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Let \( \mathbb{F} \) be an equivalence class of an even cycle matroid \( N \). We say that \( \mathbb{F} \) is row stable (resp. column stable) if for all row (resp. column) majors \( M \) of \( N \), where \( M \) has no loop and no co-loop and \( M \) is not graphic, the set of extensions of \( \mathbb{F} \) to \( M \) is an equivalence class.

2.2 A sketch of the proof of Theorem 9

We postpone the proof of the following result until Section 2.4.

**Lemma 12.** Every equivalence class of an even cycle matroid is column stable.

The following implies that if a matroid is not substantial then neither are any of its minors.

**Remark 13.** If \((G, \Sigma)\) has an intercepting pair, then so does every minor \((H, \Gamma)\) of \((G, \Sigma)\).

A signed graph \((G, \Sigma)\) is bipartite if all cycles are even. We require the following observation,

**Remark 14.** Suppose \((G, \Sigma)\) has an intercepting pair \((G_1, v_1)\) and \((G_2, v_2)\). Then there exists for \(i = 1, 2\), \(\alpha_i \subseteq \delta_{G_i}(v_i)\), such that \(\alpha_1 \triangle \alpha_2\) is a signature of \((G, \Sigma)\).

**Proof.** Every odd circuit of \((G, \Sigma)\) is a circuit of \(G_1\) using \(v_1\) or a circuit of \(G_2\) using \(v_2\). It follows that \((G, \Sigma) \setminus \left[\delta_{G_1}(v_1) \cup \delta_{G_2}(v_2)\right]\) is bipartite. Hence, there is a signature of \((G, \Sigma)\) contained in \(\delta_{G_1}(v_1) \cup \delta_{G_2}(v_2)\) and the result follows. \(\square\)

**Proof of Remark 13.** Suppose \((G, \Sigma)\) has an intercepting pair \((G_1, v_1)\) and \((G_2, v_2)\). Let \(\alpha_1 \triangle \alpha_2\) be the signature of \((G, \Sigma)\) given by Remark 14. We have \((H, \Gamma) = (G, \Sigma) \setminus I/J\) for some \(I, J \subseteq E(G)\). For \(i = 1, 2\), let \((H_i, \beta_i) := (G_i, \alpha_i) \setminus I/J\). Since, \(v_i\) is a blocking vertex of \((G_i, \alpha_i)\), there is a blocking vertex \(w_i\) of \((H_i, \beta_i)\) and we may assume that \(\beta_i \subseteq \delta_{H_i}(w_i)\). It follows from the definition of signed minor that, for some cut \(B_i\) of \(G_i\), \(\beta_i = (\alpha_i \triangle B_i) \setminus I\) and \((\alpha_i \triangle B_i) \cap J = \emptyset\). Since \(G, G_1, G_2\) are equivalent, \(B_1, B_2\) are cuts of \(G\), hence \(\alpha_1 \triangle \alpha_2 \triangle B_1 \triangle B_2\) is a signature of \((G, \Sigma)\). Hence, \(\beta_1 \triangle \beta_2\) is a signature of \((H, \Gamma)\). This proves that \((H_1, w_1)\) and \((H_2, w_2)\) is an intercepting pair of \((H, \Gamma)\). \(\square\)

We say that an equivalence class \(\mathbb{F}\) has no intercepting pair if none of the signed graphs in \(\mathbb{F}\) have an intercepting pair. Note, that we could replace “none” by “any” in the previous definition, as by definition, if a signed graph has an intercepting pair, then so does every equivalent signed graph. We postpone the proof of the following result until Section 3.
Lemma 15. Equivalence classes without intercepting pairs are row stable.

Proof of Theorem 9. Let \( N \) be a connected even cycle matroid, where none of the representations of \( N \) has an intercepting pair. Let \( M \) be a connected major of \( N \). It follows (by [1, 9]) that there exists a sequence of connected matroids \( N_1, \ldots, N_k \), where \( N = N_1 \), \( M = N_k \) and, for \( i \in [k-1] \), \( N_i+1 \) is a row or column major of \( N_i \). In particular, \( N_i \) has no loops or co-loops, for every \( i \in [k] \). Since \( N_1 \) is substantial, it is not graphic, hence neither are \( N_2, \ldots, N_k \). Let \( \mathcal{F} \) be an equivalence class of \( N \) that extends to \( M \) and, for every \( j \in [k] \), define \( \mathcal{F}_j \) to be the set of extensions of \( \mathcal{F} \) to \( N_j \). It suffices to show that, for all \( j \in [k] \), \( \mathcal{F}_j \) is an equivalence class. Let us proceed by induction. As \( N_1 = N \), the result holds for \( j = 1 \). Suppose that the result holds for \( j \in [k-1] \). By Remark 13, \( \mathcal{F}_j \) does not have an intercepting pair. Therefore, by Lemma 12 and Lemma 15, \( \mathcal{F}_j \) is column and row stable, respectively. It follows that \( \mathcal{F}_{j+1} \) is an equivalence class. \( \square \)

2.3 A sketch of the proof of Theorem 8

We say that an equivalence class \( \mathcal{F} \) has no blocking pair if none of the signed graphs in \( \mathcal{F} \) have a blocking pair. We postpone the proof of the following result until Section 5.

Lemma 16. Let \( N \) be a even cycle matroid and let \( \mathcal{F} \) be an equivalence class of \( N \) with no blocking pair. Let \( M \) be a row major of \( N \) with no loops or co-loops. Suppose that \( N \) and \( M \) are 3-connected and suppose that the set \( \mathcal{F}' \) of extensions of \( \mathcal{F} \) to \( M \) is non-empty. Then \( \mathcal{F}' \) is either an equivalence class or the union of two equivalence classes \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) without intercepting pairs.

Proof of Theorem 8. Let \( N \) be a 3-connected non-degenerate even cycle matroid. Let \( M \) be a 3-connected major of \( N \). It follows (by [10]) that there is a sequence of 3-connected matroids \( N_1, \ldots, N_k \), where \( N = N_1 \), \( M = N_k \) and, for every \( i \in [k-1] \), \( N_{i+1} \) is a row or column major of \( N_i \). In particular, \( N_i \) has no loops or co-loops for any \( i \in [k] \). Since \( N_1 \) is non-degenerate, it is not graphic, hence neither are \( N_2, \ldots, N_k \). Let \( \mathcal{F} \) be an equivalence class of \( N \) that extends to \( M \). For every \( j \in [k] \), define \( \mathcal{F}_j \) to be the set of extensions of \( \mathcal{F} \) to \( N_j \). It suffices to show that, for all \( j \in [k] \), \( \mathcal{F}_j \) is either

(a) an equivalence class, or

(b) the union of two equivalence classes without intercepting pairs.

Let us proceed by induction. As \( N_1 = N \), the result holds for \( j = 1 \). Suppose that the result holds for \( j \in [k-1] \). Consider the case where \( N_{j+1} \) is a column major of \( N_j \). If (a) holds for \( \mathcal{F}_j \), then Lemma 12
implies that (a) holds for $F_{j+1}$. If (b) holds for $F_j$, then Lemma 12 and Remark 13 imply that either (a) or (b) holds for $F_{j+1}$. Consider the case where $N_{j+1}$ is a row major of $N_j$. If (a) holds for $F_j$, then Lemma 16 implies that either (a) or (b) holds. If (b) holds for $F_j$, then Lemma 15 implies that either of (a) or (b) holds for $F_{j+1}$. □

2.4 Proof of Lemma 12

The next result, proved in [4], is an easy consequence of Theorem 1.

**Remark 17.** Suppose that $\text{ecycle}(G_1, \Sigma_1) = \text{ecycle}(G_2, \Sigma_2)$. If an odd cycle of $(G_1, \Sigma_1)$ is a cycle of $G_2$, then $G_1$ and $G_2$ are equivalent.

We say that two signed graphs $(G_1, \Sigma_1)$ and $(G_2, \Sigma_2)$ are siblings if $\text{ecycle}(G_1, \Sigma_1) = \text{ecycle}(G_2, \Sigma_2)$ and graphs $G_1$ and $G_2$ are not equivalent.

**Lemma 18.** Let $(G_1, \Sigma_1)$ and $(G_2, \Sigma_2)$ be siblings and let $\Omega \in E(G_1)$. For $i = 1, 2$, let $(H_i, \Gamma_i) := (G_i, \Sigma_i) \setminus \Omega$. Suppose that $(H_1, \Gamma_1)$ and $(H_2, \Gamma_2)$ are equivalent. Then, for $i = 1, 2$, $\Omega$ is either a bridge of $G_i$ or a signature of $(G_i, \Sigma_i)$. In particular, $\Omega$ is a co-loop of $\text{ecycle}(G_1, \Sigma_1)$.

*Proof.* We prove the statement for $i = 1$. Remark 17 implies that no odd cycle of $(G_1, \Sigma_1)$ is a cycle of $G_2$. Since $H_1$ and $H_2$ are equivalent, $\text{cycle}(H_1) = \text{cycle}(H_2)$. It follows that all odd cycles of $(G_1, \Sigma_1)$ use $\Omega$. Hence, after possibly a signature exchange, $\Sigma_1 \subseteq \{\Omega\}$. Similarly, we may assume that $\Sigma_2 \subseteq \{\Omega\}$. If $\Omega$ is a bridge of $G_1$, we are done. Suppose otherwise. If $\Sigma_1 = \emptyset$, then there exists an even cycle $C$ of $(G_1, \Sigma_1)$ using $\Omega$; hence $\Omega$ is not a bridge of $G_2$ and $\Sigma_2 \neq \{\Omega\}$. But then $\Sigma_1 = \Sigma_2 = \emptyset$ and $\text{cycle}(G_1) = \text{cycle}(G_2)$. It follows by Theorem 1 that $G_1$ and $G_2$ are equivalent, a contradiction. □

**Proof of Lemma 12.** Let $\mathbb{F}$ be an equivalence class of an even cycle matroid $N$. Let $M$ be a column extension of $N$, i.e. for some $\Omega \in E(M)$, $N = M \setminus \Omega$. Let $\mathbb{F}'$ be the set of all extensions of $N$ to $M$. Assume $M$ has no co-loops. We need to show that $\mathbb{F}'$ is an equivalence class. For otherwise there exists siblings $(G_1, \Sigma_1), (G_2, \Sigma_2) \in \mathbb{F}'$. For $i = 1, 2$, let $(H_i, \Gamma_i) := (G_i, \Sigma_i) \setminus \Omega$. Then $(H_1, \Gamma_1), (H_2, \Gamma_2) \in \mathbb{F}$. In particular, $(H_1, \Gamma_1)$ and $(H_2, \Gamma_2)$ are equivalent. Hence, by Lemma 18, $\Omega$ is a co-loop of $\text{ecycle}(G_1, \Sigma_1)$, a contradiction. □

It remains to prove Lemma 15 and 16. Lemma 15 (resp. 16) is proved in Section 3 (resp. 5).
3 Row extensions and intercepting pairs

Before we proceed with the proof of Lemma 15 we establish some preliminaries in Sections 3.1 and 3.2.

3.1 Even cut matroids

Given a graph $G$, we denote by $\text{cut}(G)$ the set of all cuts of $G$. Since the cuts of $G$ correspond to the cycles of the co-graphic matroid of $G$, we identify $\text{cut}(G)$ with that matroid and say that $G$ is a representation of that matroid.

A graft is a pair $(G, T)$ where $G$ is a graph, $T \subseteq V(G)$ and $|T|$ is even. The vertices in $T$ are the terminals of the graft. A cut $\delta(U)$ is even (respectively odd) if $|T \cap U|$ is even (respectively odd). We denote by $\text{ecut}(G, T)$ the set of all even cuts of $(G, T)$. The set $\text{ecut}(G, T)$ is the set of cycles of a binary matroid known as the even cut matroid. We identify $\text{ecut}(G, T)$ with that matroid. We say that $(G, T)$ is a representation of that matroid. Given a graph $H$, we denote by $V_{\text{odd}}(H)$ the set of vertices of $H$ of odd degree.

We will make repeated use of the following result (which was proved in [4]).

**Theorem 19.** Let $G_1$ and $G_2$ be inequivalent graphs.

1. Suppose that there exists a pair $\Sigma_1, \Sigma_2 \subseteq E(G_1)$ such that $\text{ecycle}(G_1, \Sigma_1) = \text{ecycle}(G_2, \Sigma_2)$. For $i = 1, 2$, if $(G_i, \Sigma_i)$ is bipartite define $C_i := \emptyset$; otherwise let $C_i$ be an odd cycle of $(G_i, \Sigma_i)$. Let $T_i := V_{\text{odd}}(G_i[C_{3-i}])$. Then $\text{ecut}(G_1, T_1) = \text{ecut}(G_2, T_2)$.

2. Suppose that there exists a pair $T_1 \subseteq V(G_1), T_2 \subseteq V(G_2)$ (where $|T_1|$ and $|T_2|$ are even) such that $\text{ecut}(G_1, T_1) = \text{ecut}(G_2, T_2)$. For $i = 1, 2$, if $T_i = \emptyset$ let $\Sigma_3-i = \emptyset$; otherwise let $t_i \in T_i$ and $\Sigma_3-i := \delta_{G_i}(t_i)$. Then $\text{ecycle}(G_1, \Sigma_1) = \text{ecycle}(G_2, \Sigma_2)$.

Moreover, if they exist, the pairs $\Sigma_1, \Sigma_2$ and $T_1, T_2$ are unique (up to signature exchange).

3.2 Split siblings

Consider a pair of equivalent graphs $H_1$ and $H_2$. Suppose that, for $i = 1, 2$, we have $\alpha_i \subseteq \delta_{H_i}(v_i) \cup \text{loop}(H_i)$ for some $v_i \in V(H_i)$ and that no loop is in $\alpha_1 \cap \alpha_2$. Then, for $i = 1, 2$, let $G_i$ be obtained from $H_i$ by splitting $v_i$ into $v_i^-$ and $v_i^+$ according to $\alpha_i$ and let $T_i := \{v_i^-, v_i^+\}$. As $H_1$ and $H_2$ are equivalent,
cycle(\(H_1\)) = cycle(\(H_2\)). Since cut(\(H_1\)) and cut(\(H_2\)) are the duals of cycle(\(H_1\)) and cycle(\(H_2\)) we have that cut(\(H_1\)) = cut(\(H_2\)). For \(i = 1, 2\), if \(\delta_{G_i}(U)\) is an even cut of \((G_i, T_i)\) then \(T_i \subseteq U\) or \(T_i \subseteq \bar{U}\) (because \(|T_i| = 2\)). Hence,
\[
ecut(G_1, T_1) = \text{cut}(H_1) = \text{cut}(H_2) = \text{ecut}(G_2, T_2).
\]

By Theorem 19 there is a unique pair of signatures \(\Sigma_1\) and \(\Sigma_2\) such that \(\text{ecycle}(G_1, \Sigma_1) = \text{ecycle}(G_2, \Sigma_2)\). We say that \((G_1, \Sigma_1)\) and \((G_2, \Sigma_2)\) are split siblings. Observe that, in the previous definition, if \(\Omega\) is a loop of \(H_1, H_2\) contained in \(\alpha_1 \cap \alpha_2\), then for \(i = 1, 2\), \(\Omega\) has endpoints \(T_i\) in \(G_i\). We will refer to split siblings with such an edge \(\Omega\) as \(\Omega\)-split siblings.

In light of the previous discussion, we say that a tuple \(\mathbb{T} = (H_1, v_1, \alpha_1, H_2, v_2, \alpha_2)\) is a split template if the following conditions hold:

(a) \(H_1\) and \(H_2\) are equivalent graphs;

(b) for \(i = 1, 2\), \(v_i \in V(H_i)\);

(c) for \(i = 1, 2\), \(\alpha_i \subseteq \delta_{H_i}(v_i) \cup \text{loop}(H_i)\) and no loop is in \(\alpha_1 \triangle \alpha_2\).

We say that the split siblings \((G_1, \Sigma_1)\) and \((G_2, \Sigma_2)\) or \((G'_1, \Sigma'_1)\) and \((G'_2, \Sigma'_2)\) defined in the previous paragraph arise from the split-template \(\mathbb{T}\). \(^3\)

**Remark 20.** Let \(\mathbb{T} = (H_1, v_1, \alpha_1, H_2, v_2, \alpha_2)\) be a split-template and let \((G_1, \Sigma_1)\) and \((G_2, \Sigma_2)\) be split siblings that arise from \(\mathbb{T}\). Then, up to signature exchange, we have \(\Sigma_1 = \Sigma_2 = \alpha_1 \triangle \alpha_2\).

**Proof.** For \(i = 1, 2\), vertex \(v_i\) of \(H_i\) gets split into vertices \(v^-_i\) and \(v^+_i\) of \(G_i\). We only consider the case of \(\Omega\)-split siblings as the other cases are similar. Let \(\Omega\) denote the set of loops of \(H_1, H_2\) that are in both \(\alpha_1 \cap \alpha_2\). Note, that, for \(i = 1, 2\), all edges in \(\Omega\) will have both ends in \(T_i\). By construction, \(\alpha_i \cup \Omega = \delta_{G_i}(v^-_i)\), for \(i = 1, 2\). As \(v^-_1 \in T_1\), Theorem 19 implies that \(\alpha_1 \cup \Omega\) is a signature of \((G_2, \Sigma_2)\). As \(\alpha_2 \cup \Omega\) is a cut of \(G_2\), \(\alpha_1 \triangle \alpha_2\) is a signature of \((G_2, \Sigma_2)\). By symmetry, \(\alpha_1 \triangle \alpha_2\) is also a signature of \((G_1, \Sigma_1)\). \(\Box\)

### 3.3 Proof Lemma 15

The following easy observation is the analogue to Remark 17 for the case of even cut matroids (see [4]).

---

\(^3\)Note, this is a slight departure from the definition in [5] as we do not require that \((G_1, \Sigma_1)\) and \((G_2, \Sigma_2)\) arising from \(\mathbb{T}\) to be inequivalent. This does not affect the outcome of Theorem 26 and 30 that are used in this paper.
Remark 21. Suppose that $\text{ecut}(G_1, T_1) = \text{ecut}(G_2, T_2)$. If any odd cut of $(G_1, T_1)$ is a cut of $G_2$, then $G_1$ and $G_2$ are equivalent.

Let $(G, T)$ be a graft and let $e \in E(G)$. Then $(G, T) \setminus e$ is defined as $(G \setminus e, T')$, where $T' = \emptyset$ if $e$ is an odd bridge of $(G, T)$ and $T' = T$ otherwise. $(G, T)/e$ is equal to $(G/e, T')$, where $T'$ is defined as follows.

Let $u, v$ be the ends of $e$ in $G$ and let $w$ be the vertex obtained by contracting $e$. If $x \neq w$, then $x \in T'$ if and only if $x \in T$; $w \in T'$ if and only if $|\{u, v\} \cap T| = 1$. Observe that (see [6] for instance),

Remark 22. $\text{ecut}(G, \Sigma) \setminus I/J = \text{ecut}((G, \Sigma)/I \setminus J)$.

In particular, this implies that being an even cut matroid is a minor closed property.

The following result is the analogue to Lemma 18 for even cut matroids.

Lemma 23. Suppose that $G_1$ and $G_2$ are not equivalent and $\text{ecut}(G_1, T_1) = \text{ecut}(G_2, T_2)$. Let $\Omega \in E(G_1)$.

For $i = 1, 2$, let $(H_i, R_i) := (G_i, T_i)/\Omega$. Suppose that $H_1$ and $H_2$ are equivalent. Then, for $i = 1, 2$, either $\Omega$ is a loop of $G_i$ or $|T_i| = 2$ and $T_i$ are the ends of $\Omega$ in $G_i$. In particular, $\Omega$ is a co-loop of $\text{ecut}(G_1, T_1)$.

Proof. For $i = 1, 2$, denote by $v_i$ and $w_i$ the endpoints of edge $\Omega$ in $G_i$. We prove the statement for $i = 1$. Remark 21 implies that no odd cut of $(G_1, T_1)$ is a cut of $G_2$. Since $H_1$ and $H_2$ are equivalent, $\text{cut}(H_1) = \text{cut}(H_2)$. It follows that all odd cuts of $(G_1, T_1)$ use $\Omega$. Hence, $T_1 \subseteq \{v_1, w_1\}$. Similarly, we may assume that $T_2 \subseteq \{v_2, w_2\}$. If $\Omega$ is a loop of $G_1$, we are done. Suppose otherwise. If $T_1 = \emptyset$, then there exists an even cut $B$ of $(G_1, T_1)$ using $\Omega$; hence $\Omega$ is not a loop of $G_2$ and $T_2 \neq \{v_2, w_2\}$. But then $T_1 = T_2 = \emptyset$ and $\text{cut}(G_1) = \text{cut}(G_2)$. Hence cycle$(G_1) = \text{cycle}(G_2)$ and it follows by Theorem 1 that $G_1$ and $G_2$ are equivalent, a contradiction. We conclude that $T_1 = \{v_1, w_1\}$, and symmetrically, that $|T_2| = 2$, completing the proof.

Lemma 24. Let $N$ be a non-graphic even cycle matroid and let $\mathbb{F}$ be an equivalence class of $N$. Let $M$ be a row major of $N$ with no loops or co-loops. Let $\Omega$ denote the unique element in $E(M) - E(N)$. Suppose that the set $\mathbb{F}'$ of extensions of $\mathbb{F}$ to $M$ is non-empty. Then $\mathbb{F}'$ is either an equivalence class or the union of two equivalence classes $\mathbb{F}_1$ and $\mathbb{F}_2$ and any $(G_1, \Sigma_1) \in \mathbb{F}_1$ and $(G_2, \Sigma_2) \in \mathbb{F}_2$ are $\Omega$-split siblings.

Proof. We may assume that $\mathbb{F}'$ is not an equivalence class. Hence, there exist siblings $(G_1, \Sigma_1), (G_2, \Sigma_2) \in \mathbb{F}'$. For $i = 1, 2$ let $(H_i, \Gamma_i) := (G_i, \Sigma_i)/\Omega$. By definition of $\mathbb{F}'$, $(H_1, \Gamma_1), (H_2, \Gamma_2) \in \mathbb{F}$. In particular, $H_1$ and $H_2$ are equivalent. Since $G_1$ and $G_2$ are not equivalent, Theorem 19 implies that there exists a unique pair
of terminals $T_1, T_2$ such that $\text{cut}(G_1, T_1) = \text{cut}(G_2, T_2)$. For $i = 1, 2$, we have $(H_i, R_i) = (G_i, T_i)/\Omega$ for some terminals $R_1, R_2$. Lemma 23 implies that, for $i = 1, 2$, either $\Omega$ is a loop of $G_i$ or $T_i$ are the ends of $\Omega$ in $G_i$. If the latter case occurs for both $i = 1, 2$, then we are done as $(G_1, \Sigma_1)$ and $(G_2, \Sigma_2)$ are $\Omega$-split siblings (by Theorem 19, the pair $\Sigma_1, \Sigma_2$ is uniquely determined). Now suppose that $\Omega$ is a loop of $G_i$, for $i = 1$ or $i = 2$. Then every cut of $G_i$ is a cut of $H_i$, hence a cut of $H_{3-i}$ (as $H_1$ and $H_2$ are equivalent).

It follows that every cut of $G_i$ is a cut of $G_{3-i}$. Therefore, by Remark 21, every cut of $(G_i, T_i)$ is even. Therefore $T_i = \emptyset$. It follows by Theorem 19 that $\Sigma_{3-i} = \emptyset$, in particular $M$ is graphic. Hence, $N$ is graphic as well, contradicting our hypothesis.

It remains to show that $\mathbb{F}'$ can be partitioned into exactly two equivalence classes. Suppose, for a contradiction, this is not the case. Then, there exist, for $i = 1, 2, 3$, $(G_i, \Sigma_i) \in \mathbb{F}'$, where $G_1, G_2$ and $G_3$ are pairwise inequivalent. Theorem 19 applied to the pair $G_1, G_2$ implies that there exists terminals $T_1, T_2$ such that $\text{cut}(G_1, T_1) = \text{cut}(G_2, T_2)$. Theorem 19 applied to the pair $G_2, G_3$ implies that there exist terminals $T'_2, T_3$ such that $\text{cut}(G_2, T'_2) = \text{cut}(G_3, T_3)$. Observe that the ends of $\Omega$ in $G_i$ are $T_i$. Since $\text{cut}(G_2, T_2) = \text{cut}(G_2/\Omega) = \text{cut}(G_1/\Omega) = \text{cut}(G_2, T'_2)$ we must have $T_2 = T'_2$. For $i = 1, 2$, let $v_i \in T_i$ and let $D_i := \delta_{G_i}(v_i)$. Theorem 19 applied to the pair $G_1, G_3$ implies that $D_1$ is a signature of $(G_3, \Sigma_3)$. Similarly, $D_2$ is a signature of $(G_3, \Sigma_3)$. Hence, $D_1 \triangle D_2$ is a cut of $G_3$. As $\Omega \notin D_1 \triangle D_2$ we have that $D_1 \triangle D_2$ is an even cut of $(G_3, T_3)$. It follows that $D_1 \triangle D_2$ is an even cut of $(G_1, T_1)$. Hence, $D_1 \triangle (D_1 \triangle D_2) = D_2$ is an odd cut of $G_1$. But now Remark 21 implies that $G_1$ and $G_2$ are equivalent, a contradiction. 

We are now ready for the main result of this section,

**Proof of Lemma 15.** If the result does not hold we must have $N, M, \mathbb{F}, \mathbb{F}' = \mathbb{F}_1 \cup \mathbb{F}_2$ and $\Omega$ as in Lemma 24 and $\Omega$-siblings $(G_1, \Sigma_1) \in \mathbb{F}_1$ and $(G_2, \Sigma_2) \in \mathbb{F}_2$ that arise from some template $T = (H_1, v_1, \alpha_1, H_2, v_2, \alpha_2)$. As the siblings are $\Omega$-siblings, we have $H_i = G_i/\Omega$ for $i = 1, 2$. Because of Remark 20 we may assume (after possibly a signature exchange) that $\Sigma_1 = \Sigma_2 = \alpha_1 \triangle \alpha_2$. Hence, $(H_1, \alpha_1 \triangle \alpha_2), (H_2, \alpha_1 \triangle \alpha_2) \in \mathbb{F}$. It follows that $(H_1, v_1)$ and $(H_2, v_2)$ is an intercepting pair of $(H_1, \alpha_1 \triangle \alpha_2)$, contradicting our hypothesis. 

\[16\]
4 A characterization of split siblings

We will rely on Lemma 24 to prove the missing Lemma 16. The key to the proof is a theorem in [5] that characterizes split-siblings. Before we can state the theorem we need to understand 3-connected even cycle matroids.

4.1 Connectivity

Let $G$ be a graph and let $X \subseteq E(G)$. The set $X$ is a $k$-separation of $G$ if $\min\{|X|, |\bar{X}|\} \geq k$, $|\mathcal{R}_G(X)| = k$ and both $G[X]$ and $G[\bar{X}]$ are connected. A graph $G$ is $k$-connected if it has no $r$-separations for any $r < k$.

Recall that, a signed graph $(G, \Sigma)$ is bipartite if all cycles are even.

The following was proved in [5].

Proposition 25. Suppose that $\text{ecycle}(G, \Sigma)$ is 3-connected. Then

(1) $|\text{loop}(G)| \leq 1$ and if $e \in \text{loop}(G)$ then $e \in \Sigma$;

(2) $G \setminus \text{loop}(G)$ is 2-connected;

(3) if $G$ has a 2-separation $X$, then $(G[X], \Sigma \cap X)$ and $(G[\bar{X}], \Sigma \cap \bar{X})$ are both non-bipartite.

We say that $S = (X_1, \ldots, X_k)$ is a w-sequence of $G$ if, for all $i \in [k]$, $X_i$ is a 2-separation of the graph obtained from $G$ by performing Whitney-flips on $X_1, \ldots, X_i-1$ (in this order). We denote by $W_{\text{flip}}[G, S]$ the graph obtained from $G$ by performing Whitney-flips on $X_1, \ldots, X_k$ (in this order). For our purpose the position of loops is irrelevant. Hence we will assume that loops form distinct components of the graph. Therefore, if $G$ and $G'$ are equivalent graphs that are 2-connected, except for possible loops, then $G' = W_{\text{flip}}[G, S]$ for some w-sequence $S$ of $G$.

Consider a split-template $(H_1, v_1, \alpha_1, H_2, v_2, \alpha_2)$. If $H_1$ and $H_2$ are 2-connected, except for possible loops, we have that $H_2 = W_{\text{flip}}[H_1, S]$ for some w-sequence $S$. In this case we slightly abuse terminology and say that $(H_1, v_1, \alpha_1, H_2, v_2, \alpha_2, S)$ is a split-template. By Proposition 25 we know that this occurs, for instance, if $\text{ecycle}(H_1, \Gamma_1)$ is 3-connected for some $\Gamma_1$.

4.2 Characterizing split siblings

The following was proved in [5].
Theorem 26. Let $M$ be a 3-connected even cycle matroid. Let $(G_1, \Sigma_1)$ and $(G_2, \Sigma_2)$ be representations of $M$ which are split siblings arising from a split template $(H_1, v_1, \alpha_1, H_2, v_2, \alpha_2, S)$. Then $(G_1, \Sigma_1)$ and $(G_2, \Sigma_2)$ are either simple siblings or nova siblings.

We need to define the terms “simple siblings” and “nova siblings”. We begin by defining a more restrictive notion, namely simple and nova twins.

4.2.1 Simple twins

Consider a split-template $T = (H_1, v_1, \alpha_1, H_2, v_2, \alpha_2, S)$. If $S = \emptyset$, i.e. $H_1 = H_2$, then $T$ is simple and $(G_1, \Sigma_1)$ and $(G_2, \Sigma_2)$ arising from $T$ are simple twins. By Remark 20, we may assume that $\Sigma_1 = \Sigma_2 = \alpha_1 \triangle \alpha_2$. Suppose that vertex $v_1$ of $H_1$ gets split into vertices $v_1^-$ and $v_1^+$ of $G_1$. Then $\alpha_1 \subseteq \delta_{G_1}(v_1^-)$ and $\alpha_2 \subseteq \delta_{G_1}(v_2)$. Hence, $v_1^-$ and $v_2$ form a blocking pair of $(G_1, \Sigma_1)$. Thus we have the following.

Remark 27. Simple twins have blocking pairs.

4.2.2 Nova twins

Let $(H, \Sigma)$ be a signed graph with distinct vertices $s_1$ and $s_2$. For $i = 1, 2$, let $C_i$ denote an odd circuit using $s_i$ and avoiding $s_{3-i}$. If either, $C_1$ and $C_2$ intersect in a path, or $V(C_1) \cap V(C_2) = \emptyset$ and there exists a path $P$ with ends $u_i \in V(C_i) - \{s_j\}$, for $i = 1, 2$, such that $V(P) \cap (V(C_1) \cup V(C_2)) = \{u_1, u_2\}$, then we say that $(H, \Sigma)$ contains an $\{s_1, s_2\}$-handcuff. Let $(H, \Sigma)$ be a signed graph and consider a 2-separation $X$ of $H$ where $\mathcal{B}(X) = \{s_1, s_2\}$. We say that $X$ is a handcuff-separation if $\{s_1, s_2\}$ is a blocking pair of $(H[X], \Sigma \cap X)$ and there exists an $\{s_1, s_2\}$-handcuff of $(H[X], \Sigma \cap X)$.

A family $\mathcal{S} = \{X_1, \ldots, X_k\}$ of sets of edges of a graph $H$ is a \textit{w-star} with \textit{center} $v$ if

(a) $X_i \cap X_j = \emptyset$, for all distinct $i, j \in [k]$;

(b) there exist distinct $v, w_1, \ldots, w_k \in V(H)$ such that $\mathcal{B}(X_i) = \{v, w_i\}$, for all $i \in [k]$;

(c) no edge with ends $v, w_i$ is in $X_i$, for all $i \in [k]$.

A split-template $T = (H_1, v_1, \alpha_1, H_2, v_2, \alpha_2, S)$ is nova if, for $i = 1, 2$:

(a) $S$ is a w-star of $H_i$ with center $v_i$, and

(b) all $X' \subseteq X \in \mathcal{S}$ with $\mathcal{B}_{H_i}(X') = \mathcal{B}_{H_i}(X)$ are handcuff-separations of $(H_i, \alpha_i \triangle \alpha_2)$.
We say that \((G_1, \Sigma_1)\) and \((G_2, \Sigma_2)\) arising from \(T\) are *nova twins*. This construction is illustrated in Figure 2 for the case where \(k = |S| = 2\). The signed graph on the left (resp. right) represents \((G_1, \Sigma_1)\) (resp. \((G_2, \Sigma_2)\)). The arrows indicate how each piece is flipped to obtained \((G_2, \Sigma_2)\) from \((G_1, \Sigma_1)\). Shaded regions around a vertex \(v\) indicate the odd edges incident to \(v\).

![Nova twins](image)

**Figure 2: Nova twins**

### 4.3 From twins to siblings

We say that \((G_1, \Sigma_1)\) and \((G_2, \Sigma_2)\) are simple (respectively nova) *siblings* if, for \(i = 1, 2\), there exists \((G'_i, \Sigma'_i)\) equivalent to \((G_i, \Sigma_i)\) such that \((G'_1, \Sigma'_1)\) and \((G'_2, \Sigma'_2)\) are simple (respectively nova) twins.

### 4.4 Disjoint odd circuits

**Proof of Remark 3.** Let \(T = (H_1, v_1, \alpha_1, H_2, v_2, \alpha_2, S)\) be a split-template which is nova. Let \((G_1, \Sigma_1)\) and \((G_2, \Sigma_2)\) be the siblings arising from \(T\). Because of Remark 20, we may assume that \(\Sigma_1 = \Sigma_2 = \alpha_1 \triangle \alpha_2\). Suppose that \(S = \{X_1, \ldots, X_k\}\) for some integer \(k\). Because of (b) (in the definition of nova), for every \(j \in [k]\), there exists an odd circuit \(C_j \subseteq X_j\) of \((H_1, \Sigma_1)\) avoiding \(v_1\). In particular, \(C_j\) remains an odd circuit of \((G_1, \Sigma_1)\). Thus odd circuits \(C_1, \ldots, C_k\) of \((G_1, \Sigma_1)\) are pairwise vertex disjoint. Moreover, it is easy to select \(H_1\) so that the only 2-separations of \(H_1\) are given by \(S\). Then \(G_1\) is 3-connected. Hence, (1) holds with \((G, \Sigma) = (G_1, \Sigma_1)\). Moreover, \(\text{ecycle}(G_1, \Sigma_1) = \text{ecycle}(G_2, \Sigma_2)\), thus (2) holds as required. \(\square\)
5 Row extensions and blocking pairs

5.1 Outline of the proof

The goal of this section is to prove Lemma 16. We will consider a number of results with a common set of hypothesis that we state next.

**Hypothesis 28.** \( T = (H, v, H', v', \alpha, \alpha') \) is a split template that is nova. \( G \) and \( G' \) are graphs that both contain an edge \( \Omega \) where \( H = G / \Omega \), \( H' = G' / \Omega \). In addition, \((G, \alpha \triangle \alpha')\) and \((G', \alpha \triangle \alpha')\) are nova twins arising from \( T \). Moreover,

1. \((h1)\) no signed graph equivalent to \((H, \alpha \triangle \alpha')\) has a blocking pair;
2. \((h2)\) \(ecycle(H, \alpha \triangle \alpha')\) is 3-connected;
3. \((h3)\) \(ecycle(G, \alpha \triangle \alpha')\) is 3-connected.

**Lemma 29.** If Hypothesis 28 holds, then \((G, \alpha \triangle \alpha')\) has no intercepting pair.

**Proof of Lemma 16.** We may assume by Lemma 24 that \( \mathbb{F}' \) is the union of two equivalence classes \( \mathbb{F}_1 \) and \( \mathbb{F}_2 \). Let \((G, \Sigma) \in \mathbb{F}_1\) and \((G', \Sigma') \in \mathbb{F}_2\). Again by Lemma 24, \((G, \Sigma)\) and \((G', \Sigma')\) are \( \Omega \)-split siblings, i.e. they arise from a split template \( T = (H, \alpha, v, H', v', \alpha', \Sigma) \) and \( H = G / \Omega \), \( H' = G' / \Omega \). Because of Remark 20, we may assume that \( \Sigma = \Sigma' = \alpha \triangle \alpha' \). Then \((H, \alpha \triangle \alpha') = (G, \Sigma) / \Omega \in \mathbb{F}\) and similarly \((H', \alpha \triangle \alpha') \in \mathbb{F}\).

Theorem 26 implies that \((G, \Sigma)\) and \((G', \Sigma')\) are either nova twins, or simple twins. However, the latter case does not occur for otherwise, by Remark 27, some signed graph equivalent to \((G, \Sigma)\) has a blocking pair, contradicting the hypothesis that \( \mathbb{F} \) has no blocking pairs. Note that condition \((h1)\) of Hypothesis 28 holds since \( \mathbb{F} \) has no blocking pairs. Moreover, conditions, \((h2)\) and \((h3)\) hold as \( N \) and \( M \) are 3-connected. Hence, by Lemma 29, \((G, \Sigma)\) has no intercepting pair, i.e. \( \mathbb{F}_1 \) (and similarly \( \mathbb{F}_2 \)) has no intercepting pair.

It only remains to prove Lemma 29. We say that split-templates:

\[
T = (H_1, v_1, \alpha_1, H_2, v_2, \alpha_2, \Sigma) \quad \text{and} \quad T' = (H'_1, v'_1, \alpha'_1, H'_2, v'_2, \alpha'_2, \Sigma')
\]

are *compatible* if
(a) $H_i$ and $H'_i$ are equivalent, for $i = 1, 2$, and

(b) $\alpha_i \triangle \alpha'_i$ forms a cut of $H_i$, for $i = 1, 2$.

The proof of the following result is in [5].

**Lemma 30.** Every split-template has a compatible split-template which is simple or nova.

**Lemma 31.** Suppose Hypothesis 28 holds. If $(G, \alpha \triangle \alpha')$ has an intercepting pair, then some signed graph equivalent to $(G, \alpha \triangle \alpha')$ has a handcuff-separation.

**Proof.** Suppose that, $(G, \alpha \triangle \alpha')$ has an intercepting pair $(G_1, v_1)$ and $(G_2, v_2)$. It follows from Remark 14 that we can find $\beta_1 \subseteq \delta_G(v_1) \cup \text{loop}_G(v_1)$ and $\beta_2 \subseteq \delta_G(v_2) \cup \text{loop}_G(v_2)$ such that $\beta_1 \triangle \beta_2$ is a signature of $(G, \alpha \triangle \alpha')$. As $\text{ecycle}(G, \alpha \triangle \alpha')$ is 3-connected by (h3) of Hypothesis 28, Proposition 25 implies that $G_1$ and $G_2$ are 2-connected (up to loops). Hence, $G_2 = W_{\text{mp}}[G_1, \mathbb{R}]$ for some w-sequence $\mathbb{R}$ of $G_1$.

It follows that $L = (G_1, v_1, \beta_1, G_2, v_2, \beta_2)$ is a split template as it satisfies hypotheses (a),(b),(c) (see Section 3.2).

Lemma 30 implies that there exists a split-template $L' := (G'_1, v'_1, \beta'_1, G'_2, v'_2, \beta'_2, \mathbb{R}')$ compatible with $L$ which is simple or nova. Since $L'$ is compatible with $L$, both $\beta_1 \triangle \beta'_1$ and $\beta_2 \triangle \beta'_2$ are cuts of $G_1$, hence of $G$ (as $G$ and $G_1$ are equivalent they have the same cuts). It follows that $\beta'_1 \triangle \beta'_2$ is a signature of $(G, \alpha \triangle \alpha')$. Hence, $(G'_1, \beta'_1 \triangle \beta'_2)$ is equivalent to $(G, \alpha \triangle \alpha')$. Observe that $L'$ is not simple for otherwise $\{v'_1, v'_2\}$ is a blocking pair of $(G_1, \alpha \triangle \alpha')$, contradicting the hypothesis. Hence, $L'$ is nova. It follows in particular that $(G'_1, \beta'_1 \triangle \beta'_2)$ must have a handcuff-separation.

Lemma 29 follows immediately from Lemma 31 and the following result.

**Lemma 32.** Suppose Hypothesis 28 holds. Then no signed graph equivalent to $(G, \alpha \triangle \alpha')$ has a handcuff-separation.

Hence, it only remains to prove Lemma 32. We first need preliminaries.

For a graph $H$, we say that $F = (B_1, \ldots, B_t)$ is a flower if $B_1, \ldots, B_t$ is a partition of $E(H)$ and there exist distinct vertices $u_1, \ldots, u_t$ such that

(a) $t \geq 2$ and if $t = 2$ then $|B_1| > 1$ and $|B_2| > 1,$

(b) $H[B_i]$ is connected, for every $i \in [t],$ and
For $i \in [t]$, $B_i$ is a petal with attachments $u_i$ and $u_{i+1}$.

Given a graph $H$ and $U \subseteq V(H)$ we denote by $H - U$ the graph obtained from $H$ by deleting all vertices in $U$. We write $H - u$ as shorthand for $H - \{u\}$. Let $G$ be a graph with disjoint vertex sets $A$ and $B$. An $A-B$ path is a path of $H$ with one endpoint in $A$ and one endpoint in $B$. We use “$a-b$ path” as shorthand for “$\{a\} - \{b\}$ path” and similarly, “$a-B$ path” as shorthand for “$\{a\} - B$ path”.

Lemma 33. Suppose Hypothesis 28 holds. Assume that $\mathbb{S}$ is a w-star is defined as in (a)-(c) of Section 4.2.2. Let $v^+$ and $v^-$ denote the ends of $\Omega$ in $G$ and let $X_0 := E(H) - (X_1 \cup \ldots \cup X_k)$. Then

1. $k := |S| \geq 2$;
2. for all $i \in [k]$, $\mathcal{B}_G(X_i) = \{v^-, v^+, w_i\}$;
3. for all $i \in [k]$, $(G[X_i], (\alpha \Delta \alpha') \cap X_i) \setminus \delta_G(w_i)$ is bipartite;
4. $(G[X_0], (\alpha \Delta \alpha') \cap X_0) \setminus \left(\delta_G(v^-) \cup \delta_G(v^+)\right)$ is bipartite.

Let $Z$ be a 2-separation of $G$ with $\mathcal{B}_G(Z) = \{z_1, z_2\}$ and $\Omega \notin Z$. Then, after possibly exchanging the labels of $z_1$ and $z_2$, one of the following holds:

1. $Z \subseteq X_0$, $z_1 \in \{v^-, v^+\}$ and $z_2 \in V(X_0) - \{v^-, v^+\}$;
2. There exists $i \in [k]$ such that $Z \subseteq X_i$, $z_1 = w_i$ and $z_2 \in V(X_i)$;
3. There exist $i \in [k]$ and an edge $e = (w_i, z_2) \in X_0 \cap Z$ such that $Z \subseteq X_i \cup \{e\}$.

Moreover, $z_1$ is a cut vertex of $G[X_i]$ separating $\{v^-, v^+\}$ and $w_i$;

4. $\tilde{Z} = \{\Omega, \Omega'\}$, where $\Omega' = (v^-, v^+)$ and $\Omega \notin \alpha \Delta \alpha'$, $\Omega' \in \alpha \Delta \alpha'$.

Figure 2 illustrates (1)-(4) of the previous lemma (with $\alpha = \alpha_1$ and $\alpha' = \alpha_2$). Given a graph $G$ and $X \subseteq E(G)$, we denote by $\mathcal{I}_G(X)$ the set $V_G(X) \setminus \mathcal{B}_B(X)$.

Proof of Lemma 33. Throughout this proof, unless specified otherwise, $\mathcal{B}, V, I$ mean $\mathcal{B}_G, V_G, I_G$. By the definition of w-star (2) holds and we have,

\[
\alpha \subseteq \delta_G(v^+) \cup \text{loop}(H)
\]

\[
\alpha' \subseteq \left[ \bigcup_{i \in [k]} X_i \cap \delta_G(w_i) \right] \cup \left( \left[ \delta_G(v^-) \cup \delta_G(v^+) \right] \cap X_0 \right) \cup \text{loop}(H).
\]

(\text{\textit{\textdagger}})
Note that, if \( f \in \text{loop}(H) \cap \alpha \), then \( f \) has ends \( v^- \) and \( v^+ \) in \( G \). It follows from (\(*\)) that \( k \geq 2 \), for otherwise \( S = \{X_1\} \) and \( \mathcal{B}_H(X_1) = \{v, w_1\} \), and \( \{v, w_1\} \) is a blocking pair of \((H, \alpha \triangle \alpha')\), contradicting (h1). Hence (1) holds. (3) and (4) also follow from (\(*\)). By (h2) (resp. (h3)) and Proposition 25 we know that \( H \) (resp. \( G \)) is 2-connected, up to loops. We define \( \Sigma := \alpha \triangle \alpha' \).

**Claim 1.** For any \( i \in [k] \) and \( z \in \{v^+, v^-, w_i\} \), \( G[X_i] - z \) is connected.

**Proof.** We consider the case where \( z = w_i \) only as the other cases are similar. If \( G[X_i] - w_i \) is not connected, it must have components \( G[Z_1] \) and \( G[Z_2] \). Since \( G \) is 2-connected, \( v^+ \in V(Z_1) \) and \( v^- \in V(Z_2) \). Let \( Z'_1 := Z_1 \cup \{(v^+, z) : z \in V(Z_1)\} \) and \( Z'_2 := Z_2 \cup \{(v^-, z) : z \in V(Z_2)\} \). Then \( |\mathcal{B}(Z'_1)| = |\mathcal{B}(Z'_2)| = 2 \), and for \( i = 1, 2 \), \( (G[Z'_i], \Sigma \cap Z'_i) \) is bipartite. It follows from (h3) and Proposition 25 that \( Z'_i \) and \( Z'_2 \) both consist of a single edge. Hence, \( X_i = Z'_1 \cup Z'_2 \) is not a handcuff-separation, contradicting the fact that the template \( T \) in Hypothesis 28 is nova. \( \diamond \)

**Claim 2.** Suppose that for some \( i \in [k] \), \( Z \cap X_i, \bar{Z} \cap X_i \neq \emptyset \). Then, after possibly exchanging the labels of \( z_1 \) and \( z_2 \), one of the following holds:

1. \( z_1, z_2 \in V(X_i) \) or
2. \( z_1 \in \mathcal{I}(X_i), z_2 \in \mathcal{I}(\bar{X}_i) \) and \( z_1 \) is a cut vertex of \( G[X_i] \) separating \( \{v^-, v^+\} \) and \( w_i \).

**Proof.** We may assume that (1) does not hold and \( z_2 \in \mathcal{I}(\bar{X}_i) \). Note that \( z_1 \in V(X_i) \) for otherwise, \( X_i \cup \{\Omega\} \subseteq \bar{Z} \), a contradiction. Suppose that \( z_1 \in \mathcal{B}(X_i) \). Then Claim 1 implies that \( G[X_i] - z_1 \) is connected. It follows that \( X_i - \delta_G(z_1) \) is included in \( Z \) or \( \bar{Z} \). Hence, \( X_i \) is included in \( Z \) or \( \bar{Z} \), a contradiction. Thus we may assume that \( z_1 \in \mathcal{I}(X_i) \). Suppose, for a contradiction that \( z_1 \) is not a cut vertex of \( G[X_i] \) separating \( w_i \) and \( \{v^-, v^+\} \). Then there exists a \( w_i - \{v^-, v^+\} \) path \( Q \) of \( G[X_i] - z_1 \). Because of \( Q \) and \( \Omega \), \( \{w_i, v^-, v^+\} \subseteq \mathcal{I}(\bar{Z}) \). Hence, every \( z_1 - z_2 \) path of \( G \) intersects \( \mathcal{I}(\bar{Z}) \). It follows that \( G[Z] \) is not connected, a contradiction as \( G \) is 2-connected, up to loops. \( \diamond \)

**Claim 3.** If there exists \( i \in [k] \) such that \( Z \supseteq X_i \) we have outcome (d).

**Proof.** As \( \Omega \in \bar{Z}, \Omega = \{v^-, v^+\} \) and \( \{v^-, v^+\} \in V(X_i) \) we must have that \( \{z_1, z_2\} = \{v^-, v^+\} \). Suppose for a contradiction that there exists \( p \in \mathcal{I}(\bar{Z}) \). Then \( v \) is a cut vertex of \( H \) separating \( w_i \) and \( p \), a contradiction. Hence, \( \mathcal{I}(\bar{Z}) = \emptyset \). The result follows by Proposition 25. \( \diamond \)
Claim 4. There does not exist $i_1, i_2 \in [k]$ ($i_1 \neq i_2$) such that $Z \cap X_{i_1} \neq \emptyset$, $\bar{Z} \cap X_{i_1} \neq \emptyset$ and $Z \cap X_{i_2} \neq \emptyset, \bar{Z} \cap X_{i_2} \neq \emptyset$.

Proof. Suppose for a contradiction that the claim is not true. We may assume, because of Claim 2, that $z_1 \in \mathcal{F}(X_{i_1})$ and $z_1$ is a cut vertex of $G[X_{i_1}]$ separating $\{v^-, v^+\}$ and $w_{i_1}$. Similarly, $z_2 \in \mathcal{F}(X_{i_2})$ and $z_2$ is a cut vertex of $G[X_{i_2}]$ separating $\{v^-, v^+\}$ and $w_{i_2}$. Since $\Omega = (v^+, v^-)$ and $\Omega \notin Z$, we have $v^+, v^- \in \mathcal{F}(\bar{Z})$ and $\{w_{i_1}, w_{i_2}\} \in \mathcal{F}(Z)$. Hence there is no $\{v^-, v^+\} - \{w_{i_1}, w_{i_2}\}$ path of $G - (X_{i_1} \cup X_{i_2})$.

Suppose that there exists $B \subseteq E(G) - (X_1 \cup X_2)$ such that $V(B) \cap \{v^-, v^+\} \neq \emptyset$ and $G[B]$ is connected. If there exists a vertex $b \in \mathcal{F}(B)$ then $v$ is a cut vertex of $H$ separating $b$ and $\{w_{i_1}, w_{i_2}\}$, contracting the fact that $H$ is 2-connected, up to loops. Hence, for all such $B$, $\mathcal{F}(B) = \emptyset$. Thus $B$ consists of the edge $\Omega$ and possibly another edge parallel to $\Omega$ that is in $\Sigma$. In particular, $|\Sigma| = 2$ and $G$ has a flower $F = (B \cup (X_1 \cup X_2) \cap \bar{Z}, X_1 \cap Z, X_0 - B, X_2 \cap Z)}$. Then it can be readily checked, see $(\ast)$, that after rearranging the petals (by doing Whitney-flips) we obtain a blocking pair, a contradiction to (h1).  

We may assume outcome (d) does not occur. It follows from Claim 3 that, for all $i \in [k]$, $X_i \cap \bar{Z} \neq \emptyset$. It follows now, from Claim 4, that one of the following holds,

(T1) $Z \subseteq X_0$;

(T2) $\exists i \in [k]$ such that $Z \subseteq X_i$;

(T3) $\exists i \in [k]$ such that $Z \subseteq X_i \cup X_0$, $\bar{Z} \cap X_i \neq \emptyset$, $Z \cap X_0 \neq \emptyset$.

Suppose (T1) holds. Then $z_1, z_2 \in V(X_0)$. If $\{z_1, z_2\} \cap \{v^-, v^+\} = \emptyset$ then (4) implies that $(G[Z], \Sigma \cap Z)$ is bipartite. Proposition 25 then implies that $|Z| = 1$, a contradiction. Thus we may assume that $z_1 \in \{v^-, v^+\}$. If $z_2 \in \{v^-, v^+\}$ then, $\mathcal{F}(Z) \neq \emptyset$ (because $\Omega \notin Z$) and $v$ is a cut vertex of $H$, a contradiction as $H$ is 2-connected, up to loops. We conclude that (a) holds. Suppose (T2) holds. Then $z_1, z_2 \in V(X_i)$. If $w_i \not\in \{z_1, z_2\}$ then (3) implies that $(G[Z], \Sigma \cap Z)$ is bipartite. Proposition 25 then implies that $|Z| = 1$, a contradiction. Thus we may assume that $z_1 = w_i$ and (b) holds. Suppose (T3) holds. By Claim 2, we may assume that $z_1 \in \mathcal{F}(X_i)$ and $z_1$ is a cut vertex of $G[X_i]$ separating $\{v^-, v^+\}$ and $w_i$. It follows that $Z \cap X_0$ is a 2-separation with $\mathcal{B}(Z \cap X_0) = \{w_i, z_2\}$. Note that $z_2 \notin \{v^-, v^+\}$, for otherwise $v$ is a cut vertex of $H$, a contradiction, as $H$ is 2-connected up to loops. Because of (4), $(G[Z \cap X_0], \Sigma \cap (Z \cap X_0))$ is bipartite. Proposition 25 then implies that $Z \cap X_0$ consists of a single edge $e = (w_i, z_2)$ and (c) holds.  

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Let \((G, \Sigma)\) be a signed graph and let \(X\) be a 2-separation of \(G\). We say that \(X\) is a bracelet separation if \((G[X], \Sigma \cap X)\) is not bipartite, and some \(v \in \mathcal{B}(X)\) is a blocking vertex of \((G[X], X \cap \Sigma)\). We say that a flower is maximal if no petal has a cut-vertex separating its attachments. A petal of a flower that is an edge is a petal edge.

We say that a flower \(F\) is a Type 1 flower of \((G, \Sigma)\) if

(a) \(F\) is a maximal flower;

(b) \(F\) has exactly two petals;

(c) one petal is a bracelet separation of \((G, \Sigma)\).

We say that a flower \(F\) is a Type 2 flower of \((G, \Sigma)\) if

(a) \(F\) is a maximal flower;

(b) \(F\) has exactly three petals;

(c) one petal is a bracelet separation of \((G, \Sigma)\) and one petal is a petal edge.

**Lemma 34.** Suppose Hypothesis 28 holds. Then every maximal flower of \((G, \alpha \Delta \alpha')\) is of Type 1 or Type 2.

**Proof.** Throughout the proof, we let \(\mathcal{B}\) stands stand for \(\mathcal{B}_G\). We say that a 2-separation \(Z\) of \(G\), where \(\Omega \notin Z\), is of type (a),(b),(c) or (d), respectively, if Outcome (a), (b), (c), or (d), respectively, occurs when we apply Lemma 33 to the separation \(Z\).

**Claim.** Let \(Z\) be a 2-separation of \((G, \alpha \Delta \alpha')\) where \(\Omega \notin Z\).

(1) If \(Z\) if of type (a),(b) then \(Z\) is a bracelet separation;

(2) If \(Z\) if of type (c) then \(Z\) is obtained, by adding in a flower, a petal edge to a bracelet separation;

(3) If \(Z\) is of type (d) then \(\tilde{Z}\) is a bracelet separation.

**Proof.** If \(Z\) is of type (a) then Lemma 33(4) implies that \(Z\) is a bracelet separation. If \(Z\) is of type (b) or (c) then Lemma 33(3) implies that \(Z\) is a bracelet separation. Otherwise the separation is of type (d) and (2) holds

\(\diamond\)
Let \( F = (Z_0, \ldots, Z_t) \) be a maximal flower of \((G, \alpha \Delta \alpha')\) that is not of Type 1 and not of Type 2. We may assume that \( \Omega \in Z_0 \). No two of \( Z_1, \ldots, Z_t \) are petal edges, for otherwise these petal edges are in series contradicting (h3) of Hypothesis 28. Consider first the case where exactly one of \( Z_1, \ldots, Z_t \) is not a petal edge. Then either \( t = 1 \) and \( Z_1 \) is not a petal edge, or \( t = 2 \) and exactly one of \( Z_1 \) and \( Z_2 \) is a petal edge. For \( t = 1 \), the Claim implies that \( F' \) is a Type 1 flower and for \( t = 2 \), the Claim implies that \( F' \) is a Type 2 flower, a contradiction. Thus we may assume that, for 

\[
\text{Lemma 35. Let } \hat{G} \text{ be a 2-connected graph and let } X \text{ be a 2-separation of } \hat{G}. \text{ Let } G \text{ be equivalent to } \hat{G}. \text{ Then there exists a maximal flower } F = (B_1, \ldots, B_t) \text{ of } G, \text{ such that } X \text{ is equal to the union of a subset of the petals of } G. \text{ In particular, if every maximal flower of } G \text{ has at most three petals, then every 2-separation of } \hat{G} \text{ is a 2-separation of } G.
\]

Proof. Given a graph \( H \), denote by \( c(H) \) the number of components of \( H \). Let \( M := \text{cycle}(G) = \text{cycle}(\hat{G}) \). Then for any \( X \subseteq E(\hat{G}), \) the connectivity function \( \lambda_M(X) \) satisfies the relation,

\[
\lambda_M(X) = |V_G(X) \cap V_G(\hat{X})| - c(G[X]) - c(G[\hat{X}]) + 2.
\]

Since \( X \) is a 2-separation of \( \hat{G} \), \( \lambda_M(X) = 2 \). Hence,

\[
|V_G(X) \cap V_G(\hat{X})| = c(G[X]) + c(G[\hat{X}]).
\]

Construct an auxiliary graph \( H \) as follows: the vertices of \( H \) correspond to the components of \( G[X] \) and to the components of \( G[\hat{X}] \). Join two vertices of \( H \) by \( k \) parallel edges if the corresponding components have \( k \) vertices in common. Then \( |V(H)| = |E(H)|. \) Moreover, since \( \hat{G} \) (and therefore \( G \)) is 2-connected, \( \text{deg}_H(v) \geq 2 \) for every \( v \in V(H) \) and \( H \) is connected. It follows that \( H \) consists of a circuit and the result follows.

The signed graph \( \mathcal{F}_7 \) is obtained by replacing every edge in a triangle by two parallel edges, one odd, one even, and by adding an odd loop (see Figure 3). Note that \( \text{ecycle}(\mathcal{F}_7) \) is the matroid \( F_7 \).
Lemma 36. Let \((G, \Sigma)\) be a signed graph and suppose that no signed graph equivalent to \((G, \Sigma)\) has a blocking pair. Let \(F\) be a Type 1 or Type 2 flower of \((G, \Sigma)\) and let \(Z\) be a petal of \(F\). Let \((\hat{G}, \Sigma)\) be a signed graph equivalent to \((G, \Sigma)\). Then \(Z\) is not a handcuff-separation of \((\hat{G}, \Sigma)\).

Proof. Suppose for a contradiction that \(Z\) is a handcuff-separation of \((\hat{G}, \Sigma)\). We will only prove the case of the Type 1 flowers as the proof for Type 2 is similar. Then \(F = \{Z, Z'\}\) and one of \(Z, Z'\) is a bracelet separation. Observe that \(F\) is a flower of \(\hat{G}\). Since \(|F| \leq 3\), we may assume that there exists a \(w\)-sequence \(S\) of \(G\) where \(\hat{G} = W_{np}[G, S]\) and, for all \(S \in S\), we have \(S \subseteq Z\).

Consider first the case where \(Z'\) is a bracelet separation of \((G, \Sigma)\). Then \(Z'\) remains a bracelet separation of \((\hat{G}, \Sigma)\). But then, as \(Z\) is a handcuff-separation, \(B(Z)\) is a blocking pair, contradicting our hypothesis. Consider now the case where \(Z\) is a bracelet separation of \((G, \Sigma)\). Construct a signed graph \((H, \Gamma)\) from \((G[Z], \Sigma \cap Z)\) by adding two parallel edges, one odd, one even, between vertices of \(B_G(Z)\) and by adding an odd loop. Let \(\hat{H} := W_{np}[H, S]\). Note that \((H, \Gamma)\) and \((\hat{H}, \Gamma)\) are equivalent hence they are representations of the same even cycle matroid. To obtain a contradiction we will show that \(\text{ecycle}(H, \Gamma)\) is a graphic matroid but that \(\text{ecycle}(\hat{H}, \Gamma)\) is not graphic. Since \(Z\) is a bracelet separation of \((G, \Sigma)\), \((H, \Gamma)\) has a blocking vertex. It follows from Remark 6 that \((H, \Gamma)\) is graphic. Since \(Z\) is a handcuff-separation of \((\hat{G}, \Sigma)\), \((\hat{H}, \Gamma)\) contains the signed graph \(F_7\) as a minor. As \(\text{ecycle}(F_7)\) is the matroid \(F_7\), it follows in particular that \(\text{ecycle}(\hat{H}, \Gamma)\) is not graphic.

We are now ready for the proof of the last remaining lemma.

Proof of Lemma 32. Let \((\hat{G}, \alpha \triangle \alpha')\) be a signed graph equivalent to \((G, \alpha \triangle \alpha')\). Let \(Z\) be an arbitrary 2-separation of \(\hat{G}\). Lemma 35 implies that \(Z\) is a 2-separation of \(G\). Lemma 34 implies that \(Z\) is a petal of a flower of Type 1 or Type 2 of \((G, \alpha \triangle \alpha')\). But then Lemma 36 implies that \(Z\) is not a handcuff-separation.
of $(\hat{G}, \alpha \triangleleft \alpha')$.

References


