Locally 3-arc transitive regular covers of complete bipartite graphs

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Abstract. In this paper, locally 3-arc transitive regular covers of complete bipartite graphs are studied, and results are obtained that apply to arbitrary covering transformation groups. Further results are obtained when the covering transformation group is abelian, and methods are obtained for classifying the locally 3-arc transitive regular covers of complete bipartite graphs with elementary abelian covering transformation group. In particular, the locally 3-arc transitive regular covers of complete bipartite graphs with covering transformation isomorphic to a cyclic group or an elementary abelian group of order $p^2$ are fully classified.

1. Introduction

We will assume that all graphs are finite, simple, undirected, and connected, unless otherwise stated. An $s$-arc of a graph $\Gamma$ is a sequence of vertices $(\alpha_0, \alpha_1, \ldots, \alpha_s)$ such that $\alpha_i$ is adjacent to $\alpha_{i+1}$ and $\alpha_{i-1} \neq \alpha_{i+1}$ for all possible $i$. Note that vertices can be repeated as long as $\alpha_{i-1} \neq \alpha_{i+1}$ for all possible $i$. An automorphism of the graph $\Gamma$ is a permutation of the vertices that preserves adjacency and non-adjacency. The set of all automorphisms of the graph $\Gamma$ forms a group and is denoted Aut($\Gamma$). Given a subgroup $G \leq$ Aut($\Gamma$), $\Gamma$ is $(G, s)$-arc transitive if $\Gamma$ contains an $s$-arc and any $s$-arc in $\Gamma$ can be mapped to any other $s$-arc in $\Gamma$ via an element of $G$. The graph is locally $(G, s)$-arc transitive if $\Gamma$ contains an $s$-arc and, for any vertex $\alpha$ of $\Gamma$, any $s$-arc starting at $\alpha$ can be mapped to any other $s$-arc starting at $\alpha$ via an element of $G$. In the cases that such a group $G$ exists, the graph $\Gamma$ is said to be $s$-arc transitive or locally $s$-arc transitive, respectively. Note that it is possible for a graph to be locally $(G, s)$-arc transitive but for $G$ to be intransitive on the set of vertices. (As an example, one could take the complete bipartite graph $K_{2,3}$.) On the other hand, when $\Gamma$ is locally $(G, s)$-arc transitive and every vertex in $\Gamma$ is adjacent to at least two other vertices, $G$ is transitive on the edges of $\Gamma$.

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The study of $s$-arc transitive graphs began with Tutte’s seminal studies of cubic graphs [28, 29]. Tutte proved that if $\Gamma$ is a cubic $(G,s)$-arc transitive graph, then $s \leq 5$ (and that this upper bound is the best possible). Later, Weiss demonstrated that if the valency is at least three, then $s \leq 7$ [30]. Whereas Tutte’s result used clever elementary methods, Weiss’s result relied on the Classification of Finite Simple Groups (CFSG). On the other hand, locally $s$-arc transitive graphs give geometric descriptions of rank 2 amalgams. The proof that $s \leq 9$ for any locally $s$-arc transitive graph with all vertex valencies at least three [25] relies heavily on knowledge of amalgams and the classification of weak (B,N)-pairs of rank 2 by Delgado and Stellmacher [2] and further demonstrates the deep link between the two concepts. Moreover, locally $s$-arc transitive graphs arise as incidence graphs of highly symmetric structures [18, 31] and are important in their study. For instance, the only examples of Moufang generalized octagons are the Ree-Tits generalized octagons [27], the incidence graphs of which are locally 9-arc transitive graphs.

A recent paper by Giudici, Li, and Praeger [11] set forth a program for the study of locally $s$-arc transitive graphs. First, any locally $(G,s)$-arc transitive graph that is $G$-vertex transitive is a $(G,s)$-arc transitive graph, and many have been classified under a similar program initiated by Praeger in [22]. Second, since any locally $(G,s)$-arc transitive graph of valency at least two is edge transitive, any locally $(G,s)$-arc transitive graph that is vertex intransitive must be bipartite with vertex set $V\Gamma = \Delta_1 \cup \Delta_2$ such that $G$ is transitive on each of $\Delta_1$ and $\Delta_2$, respectively. Third, there is a characterization of the graphs containing vertices of valency two [12, Theorem 6.2, Corollary 6.3]. Thus, the focus is on the graphs with minimum valency at least three that are $G$-vertex intransitive.

The breakthrough for studying these remaining $G$-vertex intransitive graphs is the normal quotient method from [11]. Let $\Gamma$ be a graph with a group of automorphisms $G$. If $G$ has a normal subgroup $N$ that acts intransitively on the vertices of $\Gamma$, define the (normal) quotient graph $\Gamma_N$ to have vertex set the $N$-orbits of vertices of $\Gamma$, and two $N$-orbits $\Sigma_1$ and $\Sigma_2$ are adjacent in $\Gamma_N$ if and only if there exist vertices $\alpha \in \Sigma_1$ and $\beta \in \Sigma_2$ such that $\alpha$ is adjacent to $\beta$ in $\Gamma$. The original graph $\Gamma$ is said to be a regular cover of $\Gamma_N$ if each $N$-orbit $\Sigma_2$ adjacent to $\Sigma_1$ contains exactly one vertex adjacent to $\alpha$ in $\Gamma$ for each $\alpha \in \Sigma_1$, and in this case $N$ is referred to as the covering transformation group. (See also Section 2 for different yet equivalent topological definitions.) Giudici, Li, and Praeger [11] showed that if $\Gamma$ is a locally $(G,s)$-arc transitive graph, then $\Gamma_N$ is a locally $(G/N,s)$-arc transitive graph.

This insight led to the following framework for studying locally $s$-arc transitive graphs:
(I) Understand the basic graphs, that is, those for which there is no nondegenerate normal quotient (a single edge would be a degenerate normal quotient, for instance).

(II) Understand the regular covers of the basic graphs.

The basic graphs will arise when the group \( G \) acting on the graph has no normal subgroup with more than two orbits. With this in mind, a group \( G \) acting transitively on a set \( \Omega \) is said to be quasiprimitive if every nontrivial normal subgroup \( N \) of \( G \) is transitive on \( \Omega \). Given the above definition of a quotient graph, it is clear that quasiprimitive groups are precisely the groups that give rise to basic graphs. The quasiprimitive groups were classified by Praeger [22] and refined into the following eight types [23]: holomorph of abelian group (HA), holomorph simple (HS), compound holomorph (HC), almost simple (AS), twisted wreath product (TW), simple diagonal (SD), compound diagonal (CD), and product action (PA). The main distinguishing factor among these different quasiprimitive actions is the socle of the quasiprimitive group \( G \), which is the subgroup \( \text{soc}(G) \) of \( G \) generated by all minimal normal subgroups of \( G \). There are four fundamentally different kinds of basic graphs for vertex intransitive locally \((G,s)\)-arc transitive graphs, where \( s \geq 2 \):

- **[Complete Bipartite]**: Complete bipartite graphs \( K_{m,n} \), where \( m \neq n \);
- **[Star Normal Quotient]**: \( G \) acts faithfully on both orbits of vertices \( \Delta_1 \) and \( \Delta_2 \), but only acts quasiprimively on \( \Delta_1 \). In this case, the quasiprimitive action of \( G \) on \( \Delta_1 \) must be one of HA, HS, AS, PA, or TW [11, Theorem 1.3];
- **[Same Type]**: \( G \) acts faithfully and quasiprimively on both orbits of vertices \( \Delta_1 \) and \( \Delta_2 \) with the same quasiprimitive type on each part, and the action of \( G \) must be one of HA, TW, AS, or PA [11, Theorem 1.2];
- **[Different Type]**: \( G \) acts faithfully and quasiprimively on both orbits of vertices \( \Delta_1 \) and \( \Delta_2 \) with a different quasiprimitive action on each part. In this case, \( G \) must act with type SD on \( \Delta_1 \) and with type PA on \( \Delta_2 \) [11, Theorem 1.2].

The graphs in [Star Normal Quotient] are covers of stars, i.e. graphs of the form \( K_{1,m} \) for some \( m > 1 \) [11, Theorem 1.1]. This is the reason these graphs are collectively referred to as those with a star normal quotient.

While progress has been made in understanding the basic graphs [8, 9, 11, 12, 13, 15, 18, 26], there has been very little work done toward understanding the regular covers of locally \( s \)-arc transitive graphs, which constitutes half of the program. The recent work [11, 17, 20, 21] on regular covers of finite symmetric graphs is limited by only considering a
single graph at a time and restricting to abelian covering transformation groups. The study of regular covers of large families of symmetric graphs collectively has only been undertaken in a few specific cases \[5, 6, 15\], even though studying regular covers of large classes of graphs is necessary to understanding symmetric graphs as a whole.

In this paper, we study the locally 3-arc transitive regular covers of complete bipartite graphs. In Section 2, we obtain results that apply to locally 3-arc transitive regular covers of complete bipartite graphs with arbitrary covering transformation group, and in Section 3 we obtain further results that apply to locally 3-arc transitive covers of complete bipartite graphs with abelian covering transformation group. Finally, we obtain results that apply to cyclic covering transformation groups and elementary abelian covering transformation groups in Sections 4 and 5, respectively, and obtain the following classification results:

**Theorem 1.1.** Let \( \Gamma = K_{m,n} \), \( m \leq n \), \( P : \tilde{\Gamma} \to \Gamma \) be a regular covering projection such that \( H := CT(P) \neq 1 \) is a cyclic group and \( G \leq \text{Aut}(\Gamma) \) lifts to \( \tilde{G} \) such that \( \tilde{\Gamma} \) is locally \((\tilde{G}, 3)\)-arc transitive. Then one of the following holds:

1. \( m = n = 2 \) and \( H \cong Z_N \), where \( N \) is any positive integer at least 2;
2. \( m = 2, n = p \) for some odd prime \( p \), and \( H \cong Z_p \);
3. \( m = n = p \) for some odd prime \( p \) and \( H \cong Z_p \).

Moreover, in each case the cover is unique (up to isomorphism).

**Theorem 1.2.** Let \( \Gamma = K_{m,n} \) with biparts of vertices \( \Delta_1 \) and \( \Delta_2 \), \( |\Delta_1| = m \leq n = |\Delta_2| \), \( P : \tilde{\Gamma} \to \Gamma \) be a regular covering projection such that \( H := CT(P) \cong Z^d_p \), \( p \) prime and \( d \geq 1 \), is an elementary abelian group and \( G \leq \text{Aut}(\Gamma) \) lifts to \( \tilde{G} \) such that \( \tilde{\Gamma} \) is locally \((\tilde{G}, 3)\)-arc transitive. Then one of the following must hold for each \( \Delta_i \):

1. \( |\Delta_i| = p^f \) for some \( f \leq d \), or
2. There exists \( X \leq \text{GL}(d,p) \) such that \( X \) has a 2-transitive action on \( |\Delta_i| \) points.

Moreover, we prove that \( K_{m,n} \) has a locally 3-arc transitive regular cover with voltages from \( H \cong Z^d_p \) only if both \( K_{2,m} \) and \( K_{2,n} \) do, and also the following:

**Theorem 1.3.** Let \( p \) be a prime and let \( K_{2,m} \) have a locally 3-arc transitive cover with covering transformation group \( H \cong Z^d_p \). For any \( f \leq d \in \mathbb{N} \), there exists a locally 3-arc transitive cover of \( K_{m,p^f} \) with covering transformation group isomorphic to \( H \).

We apply these results to covering transformation groups isomorphic to \( Z_p \times Z_p \) to obtain the following classification:

**Theorem 1.4.** Let \( H \cong Z_p \times Z_p \), where \( p \) is a prime. The only complete bipartite graphs with a locally 3-arc transitive regular cover
whose covering transformations are a subgroup of $H$ are $K_{2,2}$, $K_{2,3}$, $K_{2,p}$, $K_{2,p^2}$, $K_{3,p}$, $K_{3,p^2}$, $K_{p,p}$, $K_{p,p^2}$, and $K_{p^2,p^2}$.

**Corollary 1.5.** Let $\Gamma = K_{m,n}$, $m \leq n$, $P : \tilde{\Gamma} \to \Gamma$ be a regular covering projection such that $H := \text{CT}(P) \cong \mathbb{Z}_p \times \mathbb{Z}_p$ is an elementary abelian group of order $p^2$ and $G \leq \text{Aut}(\Gamma)$ lifts to $\tilde{G}$ such that $\tilde{\Gamma}$ is locally $(\tilde{G},3)$-arc transitive. Then one of the following holds:

1. $m = 2$ and $n = 3$;
2. $m = 2$ and $n = p^2$;
3. $m = 3$ and $n = p^2$;
4. $m = 3$ and $n = p^2$;
5. $m = p$ and $n = p^2$;
6. $m = n = p^2$.

Moreover, in each case the cover is unique (up to isomorphism).

Finally, in Section 6, we discuss how this work could be used as a starting point for classifying the locally 2-arc transitive regular covers of complete bipartite graphs.

**2. Results applicable to general covering transformation groups**

A covering projection $P : \tilde{\Gamma} \to \Gamma$ maps $V\tilde{\Gamma}$ onto $V\Gamma$, preserving adjacency, such that for any vertex $\tilde{\alpha} \in V\tilde{\Gamma}$, the set of neighbors of $\tilde{\alpha}$ is mapped bijectively onto the neighbors of $P(\tilde{\alpha})$. For a vertex $\alpha$ of $\Gamma$, the set of vertices $P^{-1}(\alpha)$ that are mapped onto $\alpha$ by $P$ is called the fiber over the vertex $\alpha$. An automorphism $g \in \text{Aut}(\Gamma)$ lifts to $\tilde{g} \in \text{Aut}(\tilde{\Gamma})$ if the following diagram commutes:

$$
\begin{array}{ccc}
\Gamma & \xrightarrow{\tilde{g}} & \tilde{\Gamma} \\
\downarrow{P} & & \downarrow{P} \\
\Gamma & \xrightarrow{g} & \Gamma
\end{array}
$$

The lift of the trivial group (identity) is known as the group of covering transformations and is denoted $\text{CT}(P)$. $\tilde{\Gamma}$ is a regular cover of $\Gamma$ if $\text{CT}(P)$ acts regularly on the set $P^{-1}(\alpha)$ for all vertices $\alpha \in V\Gamma$. (Compare these to the equivalent definitions of Section 1.) A voltage assignment is a map $\xi : \overrightarrow{E\Gamma} \to H$, where $H$ is a group, such that $\xi(\overrightarrow{\alpha\beta}) = (\xi(\overrightarrow{\beta\alpha}))^{-1}$. The derived covering graph $\tilde{\Gamma}$ of a voltage graph has vertex set $V\Gamma \times H$, where two vertices $(\alpha, h_1)$ and $(\beta, h_2)$ are adjacent iff $\alpha$ is adjacent to $\beta$ in $\Gamma$ and $h_2 = h_1\xi(\overrightarrow{\alpha\beta})$. The following theorem exhibits the deep connection between regular covers and derived covering graphs:
Theorem 2.1 ([14, Theorem 2.4.5, Section 2.5]). Every regular cover \( \tilde{\Gamma} \) of a graph \( \Gamma \) is a derived cover of a voltage graph. Moreover, if \( \tilde{\Gamma} \) is connected we may assume that the edges of a (fixed but arbitrary) spanning tree of \( \Gamma \) have been assigned the identity voltage.

Throughout this section we will assume the following: \( \Gamma \) is the complete bipartite graph \( K_{m,n} \) with biparts \( \Delta_1 \) and \( \Delta_2 \) (\( |\Delta_1| = m, |\Delta_2| = n \)), \( \Gamma \) has a regular cover \( \tilde{\Gamma} \) with covering projection \( P : \tilde{\Gamma} \to \Gamma \), and the fiber-preserving automorphisms \( \tilde{G} \) of \( \tilde{\Gamma} \) act locally 3-arc transitively on \( \tilde{\Gamma} \). We will identify the vertices of \( \Delta_1 \) with lowercase Greek letters and vertices of \( \Delta_2 \) simply with the set \( \{0, 1, 2, \ldots, n - 1\} \).

By Theorem 2.1, \( \tilde{\Gamma} \) is isomorphic to the derived graph of a voltage graph, where voltages are assigned to the co-tree edges with respect to some spanning tree \( T \) of \( \Gamma \). It does not matter which spanning tree we choose, and so we will choose the “double star” obtained by giving the identity voltage to all edges incident with either the vertex \( \alpha \in \Delta_1 \) or the vertex \( 0 \in \Delta_2 \).

We will let \( H = CT(P) \) and assume that \( H \neq 1 \) to assure that we have a nontrivial cover. Moreover, we will let \( \xi \) be the function from the edge set of \( \Gamma \) to \( H \), and for ease of notation the voltage of the edge \( -i\beta \) (directed from \( i \) to \( \beta \)) will be denoted by \( \xi_{i\beta} \).

Lemma 2.2 ([21, Corollary 7.3]). An automorphism \( g \) of \( \Gamma \) lifts to an automorphism \( \tilde{g} \) of \( \tilde{\Gamma} \) if and only if there is a solution to the equations \( \xi_{W}^{\tilde{g}} = \xi_{Wg} \), where \( W \) runs over a generating set for all closed walks based at a single vertex \( \alpha \).

Note that by Giudici, Li, Praeger [11, Theorem 1.1], if \( G = \tilde{G}/CT(P) \), then \( \Gamma \) is a locally \((G, 3)\)-arc transitive graph, and the group of automorphisms \( G \) of \( \Gamma \) lifts to \( \tilde{\Gamma} \).

At this point, we may associate each voltage \( \xi_{i\gamma} \) with the closed walk \((\alpha, i, \gamma, 0, \alpha)\) based at \( \alpha \). We note that for any fixed \( \beta \neq \alpha \in \Delta_1 \) and any \( g \in G_{\alpha\beta} \), we may use Lemma 2.2 to define a homomorphism \( \varphi_{\alpha\beta} : G_{\alpha\beta} \to \text{Aut}(H) \) by \( \xi_{i\gamma}^{\varphi_{\alpha\beta}(9)} = \xi_{i\delta\gamma} \xi_{0\beta\gamma}^{-1} \), where \( \xi_{0\delta} = 1 \) for any \( \delta \in \Delta_1 \).

Lemma 2.3. For any vertex \( \beta \neq \alpha \in \Delta_1 \), the action of the pointwise stabilizer \( G_{\alpha\beta} \) on \( \Delta_2 \) is 2-transitive.

Proof. Follows immediately from local 3-arc transitivity. \( \square \)

Lemma 2.4. For any vertex \( \beta \neq \alpha \in \Delta_1 \) and any two vertices \( i, j \neq 0 \in \Delta_2 \), \( \xi_{i\beta} \) and \( \xi_{j\beta} \) have the same order.

Proof. By Lemma 2.3, \( G_{\alpha\beta} \) acts 2-transitively on \( \Delta_2 \). Thus there exists \( g \in G_{\alpha\beta} \) such that \( 0^g = 1 \) and \( i^g = j \). Since \( g \) lifts and \( \xi_{i\beta} \) is the voltage of the closed walk \((\alpha, i, \beta, 0, \alpha)\) and \( \xi_{j\beta} \) is the voltage of the closed walk \((\alpha, j, \beta, 0, \alpha)\), by Lemma 2.2, \( \xi_{i\beta}^g = \xi_{j\beta} \), and so \( \xi_{i\beta} \) and \( \xi_{j\beta} \) have the same order. \( \square \)
Note that the argument used in Lemma 2.4 can be applied to $\Delta_1$ and some vertex $i \neq 0 \in \Delta_1$, and repeated application yields the following:

**Lemma 2.5.** The voltages assigned to the co-tree edges of $\Gamma$ must all have the same order.

**Lemma 2.6.** For any vertex $\beta \neq \alpha \in \Delta_1$ and any two vertices $i, j \neq 0 \in \Delta_2$, $\xi_{i\beta} \neq \xi_{j\beta}$.

**Proof.** By Lemma 2.3, $G_{\alpha\beta}$ acts 2-transitively on $\Delta_2$. Thus there exists $g \in G_{\alpha\beta}$ such that $0^g = i$ and $i^g = j$, and, proceeding as in Lemma 2.4, we see that $\xi_{i\beta}^g = \xi_{j\beta}^{-1}$. If $\xi_{i\beta} = \xi_{j\beta}$, then $\xi_{i\beta}$ is conjugate to the identity in Aut($\tilde{\Gamma}$), i.e. $\xi_{i\beta} = 1$. This implies that all voltages assigned to $\tilde{\Gamma}$ are the identity by Lemma 2.5, but then $H = 1$, a contradiction. □

Recall that the socle of a group $X$, denoted by $\text{soc}(X)$ is defined to be the subgroup generated by all minimal normal subgroups of $X$. Since $G_{\alpha\beta}$ is a 2-transitive group for any $\beta \neq \alpha \in \Delta_1$, by Burnside’s Theorem (see [4, Theorem 4.1B], for instance), $\text{soc}(G_{\alpha\beta})$ is either an elementary abelian $p$-group that acts regularly on $\Delta_2$ for some prime $p$ or a nonabelian simple group that acts nonregularly on $\Delta_2$. Moreover, in each case the socle is the unique minimal normal subgroup. Henceforth we will assume that $\beta \neq \alpha \in \Delta_1$ is fixed and let $S := \text{soc}(G_{\alpha\beta})$.

Now, $\text{soc}(G_{\alpha\beta})$ is the direct product of the minimal normal subgroups of $G_{\alpha\beta}$, and each minimal normal subgroup $K$ is isomorphic to a direct product $T_1 \times \ldots \times T_k$, where the $T_i$ are all simple normal subgroups of $K$ which are conjugate under $G_{\alpha\beta}$ (see [4, Theorem 4.3A]).

Now, we have a projection $\phi : G_{\alpha\beta} \to G_{\alpha\beta}^{\Delta_2}$ such that the kernel of $\phi$ is all elements of $G_{\alpha\beta}$ that act trivially on $\Delta_2$. We may restrict $\phi$ to $\text{soc}(G_{\alpha\beta})$, and it is not difficult to see that the image of $\text{soc}(G_{\alpha\beta})$ under $\phi$ is a nontrivial normal subgroup of $G_{\alpha\beta}^{\Delta_2}$. Since $S$ is the unique minimal normal subgroup of $G_{\alpha\beta}^{\Delta_2}$, $S \leq \phi(\text{soc}(G_{\alpha\beta}))$. However, each minimal normal subgroup of $G_{\alpha\beta}$ is either contained entirely in the kernel of $\phi$ or intersects the kernel trivially (since the kernel of $\phi$ is itself a normal subgroup), so $\phi(\text{soc}(G_{\alpha\beta}))$ is isomorphic to a product of minimal normal subgroups of $G_{\alpha\beta}$. Hence $S$ is isomorphic to a product $T$ of minimal normal subgroups of $G_{\alpha\beta}$. On the other hand, $G_{\alpha\beta}^{\Delta_2}$ permutes the simple direct factors of $S$, so $G_{\alpha\beta}$ permutes the simple direct factors of $T$, i.e. $T$ is a minimal normal subgroup of $G_{\alpha\beta}$. Therefore, $G_{\alpha\beta}$ has a minimal normal subgroup $T_{\alpha\beta}$ that is isomorphic to an elementary abelian $p$-group that acts regularly on $\Delta_2$ for some prime $p$ or a nonabelian simple group that acts nonregularly on $\Delta_2$.

**Theorem 2.7.** If $T = T_{\alpha\beta}$, as defined above, is a nonabelian simple group, then $T$ is isomorphic to a subgroup of $\text{Aut}(H)$, where $H = \text{CT}(P)$. 
Proof. We denote the stabilizer of the vertex 0 in $G_{\alpha\beta}$ by $G_{\alpha\beta_0}$. Note that $G_{\alpha\beta_0}$ transitively permutes the set $\{1, 2, ..., n-1\}$. Since $\xi_{i\beta}$ is the voltage of the cycle $(\alpha, i, \beta, 0, \alpha)$, for any $g \in G_{\alpha\beta_0}$ we have that $\xi_{i\beta}^g = \xi_{i\beta'}$, and so $G_{\alpha\beta_0}$ acts on the set $\{\xi_{1\beta}, ..., \xi_{(n-1)\beta}\}$, which by Lemma 2.6 is made up of distinct elements of $H$. Given $g \in G_{\alpha\beta_0}$, if its lift $\tilde{g}$ centralizes $H$, then $\xi_{i\beta}^\tilde{g} = \xi_{i\beta}$, and so $i\beta = i$ for all $i \in \{0, 1, ..., n-1\}$, which in turn implies that $g$ acts trivially on $\Delta_2$.

As noted after Lemma 2.2, there is a homomorphism $\varphi_{\alpha\beta} : G_{\alpha\beta} \to Aut(H)$. If $T$ is not isomorphic to a subgroup of $Aut(H)$, then $T$ is contained in the kernel of $\varphi_{\alpha\beta}$, i.e., the lift of $T$ must centralize $H$. However, $T$ acts nonregularly on $\Delta_2$, so there exists an element $s \in T$ that stabilizes 0 but permutes $\{1, ..., n-1\}$ nontrivially. By the above discussion, $s$ cannot centralize $H$. Therefore, $T$ cannot be in the kernel of $\varphi_{\alpha\beta}$ and must in fact be in the image of $\varphi_{\alpha\beta}$, i.e., $T \leq Aut(H)$. □

Theorem 2.8. If $T = T_{\alpha\beta}$, as defined above, is an elementary abelian $p$-group for some prime $p$, then either $T$ is isomorphic to a subgroup of $Aut(H)$, $H = CT(P)$ or the order of each nonidentity voltage assigned to $\Gamma$ is $p$.

Proof. As in Theorem 2.7, either $T$ is entirely in the kernel of $\varphi_{\alpha\beta}$, which implies that the lift of $T$ centralizes $H$, or $T$ is isomorphic to a subgroup of $Aut(H)$.

Assume the lift of $T$ centralizes $H$. By Lemma 2.5 all nonidentity voltages have the same order, say $r$. Since $T$ acts regularly on $\Delta_2$, there exists a unique $s \in T$ such that $0^s = 1$. Furthermore, $T$ is elementary abelian of order $p$, so $s$ has order $p$ and there exist vertices $a_2, a_3, ..., a_{p-1} \in \Delta_2$ such that $1^s = a_2, a_i^s = a_{i+1}$ for $2 \leq i < p-1$, and $a_{p-1}^s = 0$. Thus $(\alpha, 1\beta, 0, \alpha)^s = (\alpha, a_2, \beta, 1, \alpha)$, and so $\xi_{1\beta}^s = \xi_{a_2\beta}\xi_{1\beta}^{-1}$; on the other hand, the lift of $T$ centralizes $H$, so $\xi_{1\beta}^s = \xi_{1\beta}$, and comparing these two equations yields $\xi_{a_2\beta} = \xi_{1\beta}^s$. We may now apply $s$ to $\xi_{a_2\beta}$, and this similarly yields $\xi_{2\beta}^s = \xi_{a_2\beta} = \xi_{a_3\beta}\xi_{1\beta}^{-1}$, i.e., $\xi_{a_3\beta} = \xi_{1\beta}^3$. Proceeding inductively, we see that $\xi_{a_i\beta} = \xi_{1\beta}^i$ for $2 \leq i \leq p-1$.

Now, $G_{\alpha\beta}$ acts 2-transitively on $\Delta_2$, so there exists $h \in G_{\alpha\beta}$ such that $0^h = 1$ and $1^h = 0$. Moreover, $h^{-1} = k \in G_{\alpha\beta_0}$. Necessarily, $0^k = 0$ and $1^k = a_{p-1}$. We note that $\xi_{1\beta}^h = \xi_{1\beta}^{-1}$. On the other hand, $\xi_{1\beta}^h = \xi_{1\beta}^{k^h}$. (It should be noted here that, in general, the lift of the product of two elements of $G$ is not the product of the lifts; however, it holds in this case because the whole stabilizer $G_\alpha$ must lift, and $G_{\alpha\beta} \leq G_\alpha$.) Thus $\xi_{1\beta}^h = \xi_{1\beta}^{k^h} = (\xi_{1\beta}^k)^h = \xi_{a_{p-1}\beta} = \xi_{a_{p-1}\beta}$. Since the lift of $T$, and hence $s$, centralizes $H$. Hence $\xi_{1\beta}^{-1} = \xi_{1\beta}^{p-1}$, and $\xi_{1\beta} = 1$. Since we are assuming $H \neq 1$, the order of $\xi_{1\beta}$ is $p$, and by Lemma 2.5 the order of every nonidentity voltage is $p$. □
COROLLARY 2.9. If the lift of \( T = T_{\alpha\beta} \) centralizes the covering transformation group, then each element of an entire subgroup of order \( p^l \), where \( p^l \) is the size of the bipart \( \Delta_2 \), must be assigned as voltages to the edges incident with the vertex \( \beta \neq \alpha \in \Delta_1 \).

**Proof.** We already know from the proof of Theorem 2.8 that all powers of \( \xi \) are assigned as voltages to edges incident with \( \beta \). We will complete the proof by showing that the voltage \( \xi_{i\beta} \xi_{j\beta} \) is assigned to an edge incident with \( \beta \) whenever the voltages \( \xi_{i\beta} \) and \( \xi_{j\beta} \) are assigned to directed edges \( i\beta \) and \( j\beta \), respectively, incident with \( \beta \). Indeed, \( \xi_{j\beta}^{-1} = \xi_{i\beta}^{p-l} = \xi_{i\beta} \) for some vertex \( \ell \in \Delta_2 \). Since \( G_{\alpha\beta} \) is 2-transitive on \( \Delta_2 \) by Lemma 2.3, there exists \( g \in G_{\alpha\beta} \) such that \((\alpha, i, \beta, 0, \alpha)^g = (\alpha, i, \beta, 0, \alpha)\), which implies that \( \xi_{i\beta} = \xi_{i\beta} \xi_{j\beta} \). There exists \( s \in T \) such that \( 0^s = \ell \), and so \( gs^{-1} = k \in \Delta_{\alpha\beta} \). Hence \( \xi_{i\beta} \xi_{j\beta} = \xi_{i\beta} \xi_{j\beta} = \xi_{i\beta}^{k} = \xi_{i\beta}^{k} = \xi_{i\beta}^{k} \), since by assumption the lift of \( T \) centralizes the covering transformation group. Therefore, the voltages incident with the vertex \( \beta \) are all distinct, have order \( p \), and form a subgroup. The result follows. \( \square \)

**Proposition 2.10.** The stabilizer of a point in the 2-transitive action of \( G_{\alpha\beta}^{\Delta_2} \) is isomorphic to a section of a subgroup of \( \text{Aut}(H) \).

**Proof.** Suppose \( g \in \Delta_{\alpha\beta} \) and \( \varphi_{\alpha\beta}(g) = 1 \). Then the lift \( \tilde{g} \) must centralize \( H \). In particular, for all vertices \( i \in \Delta_2 \), \( \xi_{i\beta} = \xi_{i\beta} = \xi_{i\beta} \), i.e. for all vertices \( i \in \Delta_2 \), \( i^g = i \) (since all voltages \( \xi_{i\beta} \) are distinct by Lemma 2.6), and so \( g \) is in the kernel of \( \varphi \), where \( \varphi : G_{\alpha\beta} \to G_{\alpha\beta}^{\Delta_2} \). Hence \( \text{Ker}(\varphi_{\alpha\beta}) \leq \text{Ker}(\varphi) \), and so \( G_{\alpha\beta}^{\Delta_2} \simeq \frac{\varphi_{\alpha\beta}(G_{\alpha\beta})}{\varphi_{\alpha\beta}(\text{Ker}(\varphi))} \). \( \square \)

**Corollary 2.11.** If the lift of \( T = T_{\alpha\beta} \) does not centralize \( H \), then \( \text{Aut}(H) \) contains a subgroup that projects onto \( G_{\alpha\beta}^{\Delta_2} \).

**Proof.** Note that \( S = \text{soc}(G_{\alpha\beta}^{\Delta_2}) \) is transitive on \( \Delta_2 \), so \( G_{\alpha\beta}^{\Delta_2} = SG_{\alpha\beta}^{\Delta_2} \). Similarly, \( G_{\alpha\beta} \simeq TG_{\alpha\beta} \). Since \( \phi(T) = S \), \( S \simeq T \), and \( T \) is not in the kernel of \( \varphi_{\alpha\beta} \), using Proposition 2.10 we have:

\[
G_{\alpha\beta}^{\Delta_2} \simeq TG_{\alpha\beta}/\text{Ker}(\phi) \\
\simeq \varphi_{\alpha\beta}(T)\varphi_{\alpha\beta}(G_{\alpha\beta})/\varphi_{\alpha\beta}(\text{Ker}(\phi)) \\
= \varphi_{\alpha\beta}(G_{\alpha\beta})/\varphi_{\alpha\beta}(\text{Ker}(\phi)),
\]

and \( \varphi_{\alpha\beta}(G_{\alpha\beta}) \leq \text{Aut}(H) \), as desired. \( \square \)

### 3. Results applicable to abelian covering transformation groups

Throughout this section we will assume the following, as in Section 2: \( \Gamma \) is the complete bipartite graph \( K_{m,n} \) with biparts \( \Delta_1 \) and \( \Delta_2 \).
Similarly, \( \tilde{\Gamma} \) can be constructed to have a closed walk from \((\alpha, 1)\) to \((\alpha, h)\) giving the identity voltage to all edges incident with either the vertex \(\alpha \in \Delta_1\) or the vertex \(0 \in \Delta_2\) as our spanning tree.

**Lemma 3.1** ([12, Theorem 3.12]). Let \( \Gamma \) be a connected locally \((G, 2s)-\)arc transitive graph of valency \(\{2, k\}\), where \(k \geq 3\) and \(s \geq 1\). Then \( \Gamma \) is locally \((G, 2s + 1)-\)arc transitive.

It should be noted that the proof of this lemma in [12] is not correct; however, these issues were addressed in [12, pages 16-17] and the result still holds.

Given a regular cover \( \tilde{\Gamma} \) of \( K_{m,n} \), define \( \tilde{\Gamma}_{2,n[\alpha,\beta]} \), where \( \alpha, \beta \in \Delta_1 \), to be the graph obtained by deleting any vertex \( \gamma \) of \( \tilde{\Gamma} \) such that \( \mathcal{P}(\gamma) \notin \{\alpha, \beta\} \cup \Delta_1 \).

**Lemma 3.2.** Let \( \tilde{\Gamma} \) be a cover of \( K_{m,n} \) whose fiber-preserving automorphisms act locally 3-arc transitively on \( \tilde{\Gamma} \). Given any vertices \( \alpha, \beta, \gamma, \delta \in \Delta_1 \) such that \( \alpha \neq \beta \) and \( \gamma \neq \delta \), \( \tilde{\Gamma}_{2,n[\alpha,\beta]} \cong \tilde{\Gamma}_{2,n[\gamma,\delta]} \).

**Proof.** Assuming that \( \alpha, \beta, \) and \( \gamma \) are all distinct, the fact that \( \tilde{\Gamma}_{2,n[\alpha,\beta]} \cong \tilde{\Gamma}_{2,n[\gamma,\delta]} \) follows immediately from local 2-arc transitivity. Similarly, \( \tilde{\Gamma}_{2,n[\alpha,\gamma]} \cong \tilde{\Gamma}_{2,n[\gamma,\delta]} \), and the result follows.

By Lemma 3.2, without a loss of generality we may define \( \tilde{\Gamma}_{2,n} := \tilde{\Gamma}_{2,n[\alpha,\beta]} \) whenever \( \Gamma \) is locally 3-arc transitive. Furthermore, define \( \tilde{\Gamma}_{2,n} \) to be a connected component of \( \tilde{\Gamma}_{2,n} \).

**Proposition 3.3.** Let \( \tilde{\Gamma} \) be a cover of \( \Gamma = K_{m,n} \) whose fiber-preserving automorphisms act locally 3-arc transitively on \( \tilde{\Gamma} \). Then \( \tilde{\Gamma}_{2,n} \cong \tilde{\Sigma} \), where \( \Sigma = K_{2,n} \) and \( \tilde{\Sigma} \) is the cover of \( \Sigma \) obtained by assigning voltages to edges of \( \Sigma \) exactly as they are in the corresponding edges of \( \Gamma \).

**Proof.** Let \( H := \text{CT}(\mathcal{P}) \) and let \( K \) be the subgroup of \( H \) generated by the edges incident with \( \beta \). This implies that the fiber of each vertex in \( \tilde{\Sigma} \) has size \(|K|\). Without a loss of generality, assume that \((\alpha, 1)\) is a vertex of \( \tilde{\Gamma}_{2,n} \), and suppose that \((\alpha, h)\) is another vertex in \( \tilde{\Gamma}_{2,n} \). Thus there exists a closed walk

\[
((\alpha, 1), (i_1, 1), (\beta, \xi_{i_1}^\beta), (i_2, \xi_{i_1}^\beta \xi_{i_2}^{-\beta}), (\alpha, \xi_{i_1}^\beta \xi_{i_2}^{-\beta}) , ..., (\alpha, h))
\]

in \( \tilde{\Gamma}_{2,n} \). Note that all voltages picked up are via edges incident with \( \beta \), i.e., \( h \in K \). On the other hand, let \( k \in K \). If \( k = \xi_{i_1}^{\pm 1} \xi_{i_2}^{\pm 1} \xi_{i_3}^{\pm 1} \cdots \xi_{i_j}^{\pm 1} \), since \((\alpha, i, j, 0, \alpha)\) has voltage \( \xi_{i} \) and \((\alpha, 0, \beta, i, j, \alpha)\) has voltage \( \xi_{i}^{-1} \), it is easy to construct a closed walk from \((\alpha, 1)\) to \((\alpha, k)\). Thus the vertex
fibers in each graph have size $|K|$. Moreover, this provides the natural automorphism, since $\{(\alpha, h_1), (i, h_2)\}$ is an edge of each if and only if $h_1 = h_2$ (with a similar condition for edges incident with vertices in $\mathcal{P}^{-1}(\beta)$). The result follows. □

**Theorem 3.4.** Let $\tilde{\Gamma}$ be a regular cover of $\Gamma = K_{m,n}$ whose group of fiber-preserving automorphisms acts locally 3-arc transitively on $\tilde{\Gamma}$. Then the group of fiber-preserving automorphisms of $\hat{\Gamma}_{2,n}$ act locally 3-arc transitively on $\hat{\Gamma}_{2,n}$.

**Proof.** Let the biparts of $K_{m,n}$ be $\Delta_1 = \{\alpha, \beta, \ldots\}$ and $\Delta_2 = \{0, 1, \ldots, m-1\}$, where $\hat{\Gamma}_{2,n}$ is identified with the component of $\hat{\Gamma}_{2,n}[\alpha, \beta]$ containing the vertex $(\alpha, 1)$. First, the group of fiber-preserving automorphisms of $\tilde{\Gamma}$ is locally 3-arc transitive, so a group acting 2-transitively on $\Delta_2$ fixing both $\alpha$ and $\beta$ must lift. This group of automorphisms preserves the partition of $\Delta_1$ into $\{\alpha, \beta\}$ and $\Delta_1 \setminus \{\alpha, \beta\}$, and since the voltage assignment leading to $\hat{\Gamma}_{2,n}$ corresponds to the voltage assignment of edges of $\Gamma$ incident with $\alpha$ and $\beta$, these automorphisms will lift for $\hat{\Gamma}_{2,n}$ as well.

We next show that the automorphism $t$ of $K_{2,n}$ corresponding to $\alpha^t = \beta$, $\beta^t = \alpha$, and $i^t = i$ for all $i \in \Delta_2$ also lifts. Indeed, the equations that must be satisfied are:

$$\xi_{i\beta}^t \leftrightarrow (0, \alpha, i, \beta, 0)^t = (0, \beta, i, \alpha, 0) \leftrightarrow \xi_{i\beta}^{-1}. $$

Indeed, for any abelian group $G$ the map sending $x$ to $x^{-1}$ for all $x \in G$ is an automorphism, and so $t$ lifts as well. Hence the vertex stabilizers act 2-transitively on the neighbors, and so by [11, Lemma 3.2] the fiber-preserving automorphisms of $\hat{\Gamma}_{2,n}$ act locally 2-arc transitively. Applying Lemma 3.1 yields the result. □

**Corollary 3.5.** Let $\tilde{\Gamma}$ be a cover of $\Gamma = K_{m,n}$ whose fiber-preserving automorphisms act locally 3-arc transitively on $\tilde{\Gamma}$. If $\alpha, \beta \in \Delta_1$ are identified with the two vertices of valency $m$ in $\Sigma = K_{2,m}$ and edges of $\Sigma$ are given the same voltages as the corresponding edges in $\Gamma$, then the fiber-preserving automorphisms of the resulting cover $\hat{\Sigma}$ act locally 3-arc transitively on $\hat{\Sigma}$.

**Theorem 3.6.** Let $\Gamma = K_{m,n}$ and let $\tilde{\Gamma}$ be a locally $(\tilde{\mathcal{G}}, 3)$-arc transitive regular cover of $\Gamma$, where $\tilde{\mathcal{G}}$ is the group of fiber-preserving automorphisms of $\tilde{\Gamma}$ with abelian group of covering transformations $H = \text{CT}(\mathcal{P})$, and let $H_1$ be the subgroup of $H$ that is generated by all the voltages incident with a single vertex $\beta \in \Delta_1$ (that is not one
(of the two vertices not of valency 1 in the double star spanning tree). Then $H \lesssim H_1^{m-1}$.

**Proof.** Let $\alpha \in \Delta_1$ be the vertex in $\Delta_1$ such that every edge incident with $\alpha$ is assigned the identity voltage. By Corollary 3.5, for any $\beta \neq \alpha \in \Delta_1$, the group of voltages incident with $\beta$ is isomorphic to $H_1$. Since there are $m - 1$ vertices different from $\alpha$ in $\Delta_1$, $H$ is isomorphic to a section of $H_1^{m-1}$, where the particular section is determined by relations between voltages all incident with a single vertex in $\Delta_2$. However, since $H_1^{m-1}$ is abelian, any section is isomorphic to a subgroup, and the result holds. □

Theorem 3.4, Corollary 3.5, and Theorem 3.6 show that the search for locally 3-arc transitive regular covers $K_{m,n}$ with abelian covering transformation group comes down to finding the locally 3-arc transitive covers of $K_{2,m}$ and $K_{2,n}$ by abelian groups. Once these are known, the possible voltage groups for $K_{m,n}$ are known, and the problem reduces to seeing which actually work.

4. Locally 3-arc transitive regular covers of complete bipartite graphs with cyclic covering transformation group

In this section, we will assume that the group of covering transformations is cyclic, i.e. let $H = CT(P) = \langle \xi \rangle \cong \mathbb{Z}_N$ for some integer $N \in \mathbb{N}$, $N > 1$, where the cyclic group is viewed multiplicatively.

As in the previous section, we fix $\beta \neq \alpha \in \Delta_1$, and let $T = T_{\alpha \beta}$ be the minimal normal subgroup of $G_{\alpha \beta}$ that projects onto $\text{soc}(G_{\alpha \beta})$. Note that $T$ is an elementary abelian $p$-group that acts regularly on $\Delta_2$ for some prime $p$ or a nonabelian simple group that acts nonregularly on $\Delta_2$.

**Lemma 4.1.** If $|T| > 2$, then the lift of $T$ must centralize $H$.

**Proof.** Suppose not. Then, by Corollary 2.11, $\text{Aut}(H)$ contains a subgroup with a 2-transitive action. However, $H$ is cyclic, and so $\text{Aut}(H)$ is also cyclic, a contradiction. □

**Lemma 4.2.** If $|T| > 2$, then $G_{\alpha \beta}^{\Delta_2} \cong \mathbb{Z}_p \times \mathbb{Z}_{p-1}$ and $H \cong \mathbb{Z}_p$.

**Proof.** By Lemma 4.1, the lift of $T$ centralizes $H$, i.e. $T$ is contained in the kernel of $\varphi_{\alpha \beta}$. By Propositions 2.7 and 2.8, $T$ is elementary abelian $p$-group and $H$ must be isomorphic to $\mathbb{Z}_p$. Since $\text{soc}(G_{\alpha \beta}) \cong T$ and is elementary abelian, $|\Delta_2| = |T| = p^d$ for some integer $d > 1$. However, by Lemma 2.6, each voltage $\xi_{1, \beta}, \ldots, \xi_{(p^d - 1), \beta}$ must be a distinct element of $H \cong \mathbb{Z}_p$, and so $d = 1$. □

**Proposition 4.3.** The only possibilities for $K_{m,n}$ are $K_{2,p}$ and $K_{p,p}$, where $p$ is a prime.
Proof. By Lemma 4.2 we may assume that \( n = p \) for some prime \( p \). On the other hand, using the same reasoning, \( |\Delta_1| = p' \) for some prime \( p' \). If both \( p, p' > 2 \), then by Lemma 4.2 it is impossible for all edges to have voltages from both \( Z_p \) and \( Z_{p'} \) unless \( p = p' \). Hence the only possibilities are \( K_{2,p} \) and \( K_{p,p} \).

Proof of Theorem 1.1 First, we note that Proposition 4.3 implies that \( K_{2,2}, K_{2,p}, \) and \( K_{p,p} \) are the only three possibilities; all that remains is to show that unique covers actually exist in each listed case.

We note that \( K_{2,2} \) is a four cycle, and the cover of \( K_{2,2} \) derived from the group \( Z_N \) is a 4N-cycle, which is unique and satisfies all requirements.

Suppose now that \( m = 2 \) and \( n = p \), where \( p \) is an odd prime. By Lemma 4.2, \( N = p \). Continuing with the notation above, let \( \Delta_1 = \{ \alpha, \beta \} \) and \( \Delta_2 = \{ 0, 1, ..., p-1 \} \). Since the automorphism group of \( K_{2,p} \) is transitive on all co-tree edges incident with \( \beta \), up to isomorphism there is only one way to assign voltages to the edges of \( K_{2,p} \) incident with \( \beta \), and so there is at most one such cover (up to isomorphism). Let \( Z_p = \langle \xi \rangle \) and without a loss of generality assume that \( \xi_{i,\beta} = \xi^i \) for all \( i \). Note that the closed walks based at the vertex \( \alpha \) are generated by the closed walks of the form \( (\alpha, i, \beta, 0, \alpha) \), \( 1 \leq i \leq p - 1 \). Thus by Lemma 2.2 an automorphism \( g \) of \( \Gamma = K_{2,p} \) lifts to \( \bar{g} \) if and only if there exists a solution to the equations \( \xi_{i,\beta}^p = \xi_{W_i}^p \) for all \( i \), where \( W_i = (\alpha, i, \beta, 0, \alpha) \).

Given a primitive element \( t \) of the field \( GF(p) \), let \( g \) be the \((p-1)\)-cycle \((1 \ 2 \ ... \ t^{p-2})\) in \( Sym(\Delta_2) \), and assume that \( g \) fixes both \( \alpha \) and \( \beta \). Then, for all \( i > 0 \), assuming that \( i \equiv t^k_i \) (mod \( p \)), \( \xi_{i,\beta}^p = \xi_{i,\beta}^{t^k_i} = (\xi^{t^k_i})^\beta \); on the other hand, \( \xi_{W_i}^p = \xi_{(t^k_i)^p,\beta} = \xi^{(t^k_i)^p} = \xi^{t^k_i+1} \). Indeed, these equations are satisfied by the outer automorphism \( \phi \) of \( H \) given by \( \phi : \xi \mapsto \xi^t \), and so the \((p-1)\)-cycle \( g \) lifts.

Let \( h \) be the \( p \)-cycle \((0 \ 1 \ ... \ (p-1))\) in \( Sym(\Delta_2) \). Then \( \xi_{h,\beta}^i = (\xi^i)^\beta \); on the other hand, \( \xi_{W_i}^h = \xi_{(i+1),\beta} \xi_{i,\beta}^{t^k_i} = \xi^{t^k_i+1} = \xi^{t^k_i} \). These equations are easily satisfied by an element in the centralizer of \( \xi \), and so \( h \) lifts. The group generated by \( h \) and \( g \) acts 2-transitively on \( \Delta_2 \), as desired.

Next, let \( x \) be the automorphism of \( K_{2,p} \) that interchanges \( \alpha \) and \( \beta \) but fixes all other vertices. We will now associate the voltage \( \xi_{i,\beta} \) with the closed walk \( U_i := (0, \alpha, i, \beta, 0) \) based at 0. In order for \( x \) to lift, the group equations \( \xi_{i,\beta}^x = \xi_{U_i} = \xi_{i,\beta}^{-1} \) must have a solution. However, this corresponds to the outer automorphism \( \zeta : \xi \mapsto \xi^{-1} \), and so \( x \) lifts as well. Therefore, the cover of \( K_{2,p} \) satisfies all the desired properties.

Finally, we suppose that \( m = n = p \), where \( p \) is an odd prime. By Lemma 4.2 once again \( N = p \). Let \( \Delta_1 = \{ \alpha_0, \alpha_1, ..., \alpha_{p-1} \} \), let \( \Delta_2 = \{ 0, 1, ..., p-1 \} \), and assume that the spanning tree with identity voltages consists of all edges incident with either \( \alpha_0 \) or 0. We will
assign voltages to co-tree edges by letting \( \xi_{\alpha\beta} \) be the \((i,j)\)-entry of the following \((p - 1) \times (p - 1)\) matrix:

\[
\begin{pmatrix}
    \xi & \xi^2 & \xi^3 & \ldots & \xi^{p-1} \\
    \xi^2 & (\xi^2)^2 & (\xi^2)^3 & \ldots & (\xi^2)^{p-1} \\
    \xi^3 & (\xi^3)^2 & (\xi^3)^3 & \ldots & (\xi^3)^{p-1} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    \xi^{p-1} & (\xi^{p-1})^2 & (\xi^{p-1})^3 & \ldots & (\xi^{p-1})^{p-1}
\end{pmatrix}
\]

It is clear that each row and each column of the matrix contains each nonidentity element of \( \langle \xi \rangle \) exactly once. Hence we have a voltage assignment which satisfies all the above lemmas, and, up to permuting the vertices of \( K_{p,p} \), this assignment is unique. Arguing again as in the \( K_{2,p} \) case, we see that the desired automorphisms lift, and so we again get a cover of the desired type.

\[\square\]

5. Locally 3-arc transitive regular covers of complete bipartite graphs with elementary abelian covering transformation group

In this section, we will assume that the group of covering transformations is elementary abelian; that is, we will assume that \( H := \text{CT}(\mathcal{P}) \cong \mathbb{Z}_d^p \), where \( p \) is a prime and \( d \geq 2 \) (the case \( d = 1 \) falls under the cyclic case).

**Proposition 5.1.** Let \( K_{m,n} \) have a locally \((\tilde{G},3)\)-arc transitive cover with group of covering transformations \( H \), where \( G \leq \text{Aut}(K_{m,n}) \), and let the biparts of \( K_{m,n} \) be \( \Delta_1 \) and \( \Delta_2 \) with \( \alpha, \beta \in \Delta_1 \). If \( H \cong \mathbb{Z}_d^p \) is elementary abelian, then the stabilizer of a point in the 2-transitive action of \( G_{\alpha\beta}^\Delta \) is isomorphic to a subgroup of \( \text{Aut}(H) \cong \text{GL}(d,p) \).

**Proof.** By Proposition 2.10, we know that \( G_{\alpha\beta}^\Delta \) is isomorphic to a section of \( \text{Aut}(H) \). Identify \( \Delta_2 \) with the set \( \{0, 1, \ldots, n-1\} \) and assume that all edges incident with \( \alpha \) and 0 in \( K_{m,n} \) have been assigned the identity voltage. To see that \( G_{\alpha\beta}^\Delta \) is actually isomorphic to a subgroup of \( \text{Aut}(H) \), we first note that the subgroup of \( H \) generated by the voltages assigned to edges of \( K_{m,n} \) incident with \( \beta \), which we will call \( N_\beta \), is normal in \( H \), since \( H \) is abelian. As noted in Section 2, we may define a faithful action of \( G_{\alpha\beta}^\Delta \) on \( K \) via \( \xi_{\gamma\beta}^g = \xi_{\gamma\beta} \) for all \( g \in G_{\alpha\beta}^\Delta \). To extend this action to an action on \( H \), we note that there exists \( N' \leq H \) such that \( H \cong N_\beta \times N' \), and we define \( x^g := x \) for all \( x \in N' \). Therefore, \( G_{\alpha\beta}^\Delta \cong \text{GL}(d,p) \), as desired.

**Proof of Theorem 1.2** This follows from Theorems 2.7 and 2.8 and Proposition 5.1.

By Theorem 3.6, the process of finding the locally 3-arc transitive covers comes down to finding locally 3-arc transitive covers of \( K_{2,m} \) and
\[ K_{2,n}. \] While it can be extremely difficult to determine which \( K_{2,m} \) have lifts in full generality, there are certain cases that are not so difficult.

**Proposition 5.2.** Let \( H \cong \mathbb{Z}_p^d \). Then \( \Gamma = K_{2,p^f} \) has a locally 3-arc transitive regular cover with \( CT(\mathcal{P}) \leq H \) if and only if \( 1 \leq f \leq d \).

**Proof.** First, note that when \( f > d \) there are not enough distinct elements of \( H \) to assign to edges of \( \Gamma \), so \( 1 \leq f \leq d \).

Let the biparts of \( \Gamma \) be \( \Delta_1 = \{\alpha, \beta\} \) and \( \Delta_2 = \{0, 1, \ldots, p^f - 1\} \). Define \( W_i \) to be the closed walk \( (\alpha, i, \beta, 0, \alpha), 1 \leq i \leq p^f - 1 \). The \( W_i \) form a generating set for all closed walks based at \( \alpha \). Choose \( K \leq H \) such that \( |K| = p^f \), and define \( \xi_{i\beta} \) to be a different nonidentity element of \( K \) for \( 1 \leq i \leq p^f - 1 \). Let \( \tilde{\Gamma} \) be the resulting regular cover. First, let \( h \in \text{Aut}(\Gamma) \) be the automorphism that fixes \( \Delta_2 \) but interchanges \( \alpha \) and \( \beta \). Thus

\[ \xi_{W_i^h} \leftrightarrow (\alpha, i, \beta, 0, \alpha)^h = (\beta, i, \alpha, 0, \beta) \leftrightarrow \xi_{i\beta}^{-1} = \xi_{W_i}^{-1}. \]

Since there is always an automorphism inverting all elements of an abelian group, \( h \) lifts.

We may identify \( \{0, \ldots, p^f - 1\} \) with the elements of \( K \cong \mathbb{Z}_p^f \). Viewing \( K \) additively as a vector space, we may label the elements of \( K \) as \( 0, 1, \ldots, (p^f - 1) \), where \( \overrightarrow{i} \) is the voltage of the edge \( i\beta \). Now, the \( W_i \) generate a set of closed walks based at \( \alpha \), and the voltage of \( W_i \) is \( \overrightarrow{i} \).

Suppose \( g \in \text{Aut}(\Gamma) \) such that \( g \) fixes \( \alpha, \beta, \) and 0. Then

\[ \xi_{W_i^g} \leftrightarrow (\alpha, i, \beta, 0, \alpha)^g = (\alpha, i^g, \beta, 0, \alpha) \leftrightarrow \overrightarrow{i^g}. \]

Thus any \( g \in \text{Aut}(\Gamma) \) that acts on \( \{1, \ldots, p^f - 1\} \) as an element of \( \text{GL}(f, p) \) acts on \( \{0, 1, \ldots, (p^f - 1)\} \) will lift to an automorphism of \( \tilde{\Gamma} \).

Now, let \( g_{ij}, 0 \leq i, j \leq p^f - 1 \), be defined to be the permutation of \( \{0, 1, \ldots, p^f - 1\} \) such that \( i^{g_{ij}} = i \) and \( j^{g_{ij}} = k \), where \( \xi_{kj} \) has voltage corresponding to \( \overrightarrow{j} + \overrightarrow{k} \). Since the voltages on edges incident with \( \beta \) are all distinct, this is well-defined. Hence

\[ \xi_{W_j^{g_{ij}}} \leftrightarrow (\alpha, j, \beta, 0, \alpha)^{g_{ij}} = (\alpha, k, \beta, i, \alpha) \leftrightarrow \overrightarrow{j} + \overrightarrow{i} - \overrightarrow{i} = \overrightarrow{j} = \xi_{W_i}. \]

Since \( \xi_{W_i}^{g_{ij}} = \xi_{W_j} \) is satisfied for all \( j \) by an element centralizing \( H, g_i \) lifts. Therefore, a group of automorphisms of \( \Gamma \) stabilizing \( \Delta_1 \) that acts isomorphically to \( AGL(f, p) \) on \( \Delta_2 \) lifts, and so \( \tilde{\Gamma} \) is locally 3-arc transitive, as desired. \( \Box \)

Combined with Theorem [1.2] Proposition 5.2 shows that finding locally 3-arc transitive regular covers of \( K_{m,n} \) with elementary abelian covering transformations isomorphic to \( \mathbb{Z}_p^d \) comes down to representation theory over \( GF(p) \). We will now use these results to classify the complete bipartite graphs with locally 3-arc transitive regular covers whose covering transformation group is a subgroup of \( \mathbb{Z}_p \times \mathbb{Z}_p \), where \( p \) is a prime.
Lemma 5.3. Let \( p, r \) be distinct primes. If \( Z_r^f \subseteq GL(d, p) \), then \( f \leq d \).

Proof. This is a basic exercise in representation theory and is left to the reader. □

Theorem 5.4. Let \( H \cong Z_p \times Z_p \), where \( p \) is a prime. The only values of \( m \) for which the complete bipartite graph \( K_{2,m} \) has a locally 3-arc transitive regular cover whose covering transformations are a subgroup of \( H \) are \( m = 2, 3, p, \) and \( p^2 \).

Proof. Let the biparts of \( \Gamma = K_{2,m} \) be \( \Delta_1 = \{ \alpha, \beta \} \) and \( \Delta_2 \). Let \( G \leq \text{Aut}(\Gamma) \) be the subgroup of automorphisms that lifts to \( \tilde{\Gamma} \) such that \( \tilde{\Gamma} \) is locally \((\tilde{\Gamma}, 3)\)-arc transitive. By Corollary Theorem 1.2 if \( S := \text{soc}(G_{\alpha \beta}^2) \), then either \( S \cong Z_p^f \) for some \( f \in \mathbb{N} \), or \( G_{\alpha \beta}^2 \not\subseteq GL(2, p) \). By Proposition 5.2, if \( S \cong Z_p^f \) then \( f = 1, 2 \), and both \( K_{2,p} \) have \( K_{2,p^2} \) have such covers. We also know that \( K_{2,2} \) has such a cover by Theorem 1.1.

Assume now that \( G_{\alpha \beta}^2 \not\subseteq GL(2, p) \). By [17, Proposition 5.5.10], no nonabelian finite simple group has a nontrivial representation of degree 2 in odd characteristic, and since \( |GL(2, 2)| = 6 \), no nonabelian finite simple group is isomorphic to a subgroup of \( GL(2, 2) \). Hence \( S \cong Z_r^f \) for some prime \( r \neq p \), and by Lemma 5.3 \( f \leq 2 \). We may assume further that \( r \geq 3 \).

Suppose first that \( f = 2 \). It is not hard to show that \( S \) must be in the kernel of the determinant map from \( GL(2, p) \) to \( GF(p)^* \), so \( S \not\subseteq SL(2, p) \). Moreover, the center of \( SL(2, p) \) has order 2, so either \( S \cong Z_2^2 \) or \( S \not\subseteq PSL(2, p) \). However, the only subgroups of \( PSL(2, p) \) isomorphic to \( Z_r^2 \) for \( r \neq p \) are isomorphic to \( Z_r^2 \) (see, for instance, [3]). Hence \( S \cong Z_2^2 \), and so \( G_{\alpha \beta}^2 \cong A_4 \). However, \( A_4 \) has no faithful degree 2 representation, a contradiction. Hence \( f = 1 \), and \( G_{\alpha \beta}^2 \cong Z_r : Z_{r-1} \). However, when \( r - 1 > 2 \) the group \( Z_r : Z_{r-1} \) has no faithful representation of degree 2 (see, for instance, [16, Theorem 25.10]), so the only remaining possibility is \( r = 3 \). Since \( GL(2, 2) \cong S_3 \) and, when \( p > 2 \), \( \langle \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rangle \cong S_3, S_3 \not\subseteq GL(2, p) \) for all primes \( p \).

All that remains to show is that \( K_{2,3} \) actually has a locally 3-arc transitive regular cover with covering transformation group isomorphic to \( Z_p \times Z_p \). Let \( H = \langle \xi, \xi' \rangle \cong Z_p \times Z_p \). If each of \( \xi, \xi' \) is assigned to a different co-tree edge of \( K_{2,3} \), it is not hard to show that the entire automorphism group of \( K_{2,3} \) lifts, and the computations are left to the reader. (See also Theorem 5.9 below.) □

Corollary 5.5. \( K_{2,3} \) has a unique locally 3-arc transitive regular cover with covering transformation group isomorphic to \( Z_p \times Z_p \), where \( p \) is an odd prime.
Proof. There are only two co-tree edges of $K_{2,3}$, so each must be given a generator of $Z_p \times Z_p$ as a voltage. This assignment is unique up to isomorphism, and, as noted in the proof of Theorem 5.4, the entire automorphism group of $K_{2,3}$ lifts in this case.

We note at this point that the element $t \in \text{Aut}(K_{2,m})$ that stabilizes $\Delta_2$, where $|\Delta_2| = m$, and swaps the other two vertices always lifts when the covering transformation group is abelian. (See the proof of Theorem 5.4.) Given this observation, we may assume that the group of automorphisms that lifts from $\text{Aut}(K_{2,m})$ is isomorphic to $M \times Z_2$, where $M$ acts trivially on $\Delta_1$ and the element of order two acts as $t$ acts.

Proposition 5.6. Suppose that $K_{2,m}$ and $K_{2,n}$ each have locally 3-arc transitive covers with voltages from an elementary abelian group $H$, and suppose that the groups that lift are $M \times Z_2$ and $N \times Z_2$, respectively. Then there exists a voltage assignment with elements from $H$ to edges of $K_{m,n}$ such that $M_0 \times N_0$ lifts, where $M_0$ and $N_0$ are the stabilizers of a vertex in $M$ and $N$, respectively.

Proof. Assume that there is a way to assign voltages of $H$ to edges of $K_{2,m}$ and $K_{2,n}$, respectively, such that the resulting derived covering graph is locally 3-arc transitive. We will assume further that the vertices in $K_{2,m}$ have been labeled $\{\alpha, \beta\}$ in the bipart of size two and $\{\gamma_0, \ldots, \gamma_{m-1}\}$ in the bipart of size $m$, and assume that the pointwise stabilizer of $\alpha$ and $\beta$ (that acts 2-transitively on $\{\gamma_0, \ldots, \gamma_{m-1}\}$) that lifts is $M$. Similarly, we will assume that the vertices in $K_{2,n}$ have been labeled $\{\alpha, \beta\}$ in the bipart of size two and $\{\delta_0, \ldots, \delta_{n-1}\}$ in the bipart of size $n$, and assume that the pointwise stabilizer of $\alpha$ and $\beta$ (that acts 2-transitively on $\{\delta_0, \ldots, \delta_{n-1}\}$) that lifts is $N$. Without a loss of generality, we may assume that identity voltages were assigned to all edges incident with $\alpha$ and $\gamma_0$ (respectively $\delta_0$), and furthermore we may assume that $\xi_{\gamma_1 \beta} = \xi_{\delta_1 \beta}^{-1} = x \in H$.

We will now show that there exists a voltage assignment for edges of $K_{m,n}$ such that a group isomorphic to $M_0 \times N_0$ lifts. Label the vertices of $K_{m,n}$ as $\{\gamma_0, \ldots, \gamma_{m-1}\}$ in the bipart $\Delta_1$ of size $m$ and $\{\delta_0, \ldots, \delta_{n-1}\}$ in the bipart $\Delta_2$ of size $n$. Let $G \cong M_0 \times N_0$ with subgroups $M_0', N_0' \leq G$, where $M_0'$ acts trivially on $\Delta_2$ but acts as $M_0$ acts on $\Delta_1$, and $N_0'$ acts trivially on $\Delta_1$ but acts on $\Delta_2$ as $N_0$ does. Note that every element of $g \in G$ can be written uniquely as $g = cd$, where $c \in M_0', d \in N_0'$, and that elements of $M_0', N_0'$ commute. Fix elements $c_j$ in $M_0$, $1 \leq j \leq m - 1$, such that for each $j$, $\gamma_0^{c_j} = \gamma_0$ and $\gamma_1^{c_j} = \gamma_j$.

Similarly, fix elements $d_j$ in $N_0$, $1 \leq j \leq n - 1$, such that for each $j$, $\delta_0^{d_j} = \delta_0$ and $\delta_1^{d_j} = \delta_j$. By Lemma 2.2, for each $c \in M_0$ there is a solution to the system of equations $\xi_{c_{\gamma_i \beta}} = \xi_{W_i}$, where $W_i$ is the closed walk $(\alpha, \gamma_i, \beta, \gamma_0, \alpha)$, and for each $d \in N_0$ there is a solution to the system
Moreover, by Proposition 5.1 for each $\xi$ of equations $18$ ERIC SWARTZ $\in d$ of $M$ that the regular cover. Define closed walks $W^c$ and $\delta$ of $M$ write $g$ $K$ of $c$ we note that each hypothesis.

Note that, by the definition of lift, $c$ and so every to an automorphism $\tilde{c}$ this way, we see that $\xi$ $G$ $cd$ and so every $\tilde{c}$ $\tilde{d}$ of $\tilde{c}$ $\tilde{c}$ $\tilde{d}$ act on $\tilde{c}$. Therefore, let $\tilde{\Gamma}$ be the resulting regular cover. Define closed walks $W_{ij} := (\gamma_0, \delta_0, \gamma_1, \delta_j, \gamma_0)$. We note that the $W_{ij}$ form a generating set for all closed walks based at $\gamma_0$. We will first show that all elements $d_k \in N'_0$ lift. Indeed,

$$\xi_{W_{ij}}^{d_k} \leftrightarrow (\gamma_0, \delta_0, \gamma_1, \delta_j, \gamma_0)^{d_k}$$

$$= ((\gamma_0, \delta_0, \gamma_1, \delta_j^d, \gamma_0)$$

$$\leftrightarrow ((x^{\hat{c}})^{d_j})^{d_k}$$

$$= (\xi_{W_{ij}})^{d_k}.$$ 

Hence we may define $\xi_{W_{ij}}^d := \xi_{W_{ij}}^d$ for all $d \in N'_0, \xi \in H$, and since by assumption the action of $\tilde{d}_k$ on $H$ is well-defined, by Lemma 2.2 $d$ lifts to an automorphism $\tilde{d}$ of $\tilde{\Gamma}$. Similarly, $\xi_{W_{ij}} = (\xi_{W_{ij}})^{\tilde{c}}$ for all $c \in M'_0$, and so every $c$ in $M'_0$ lifts to an automorphism $\tilde{c}$ of $\tilde{\Gamma}$ as well.

Finally, let $g$ be any element of $G$. It is easy to see that we may write $g$ uniquely as $g = cd$ for $c \in M'_0, d \in N'_0$. Note that

$$\xi_{W_{ij}}^{cd} = (\xi_{W_{ij}})^{cd} = (\xi_{W_{ij}})^d$$

and similarly $\xi_{W_{ij}}^{cd} = (\xi_{W_{ij}})^{cd}$. Since each $g \in G$ is written uniquely this way, we see that $\tilde{c}d$ will lift if and only if $\xi_{W_{ij}}^{\tilde{c}d} = (\xi_{W_{ij}})^{\tilde{c}d}$ for all $i, j$. Note that, by the definition of lift,

$$\tilde{c}d^{-1} \tilde{d}^{-1} \circ \mathcal{P} = \mathcal{P} \circ cd^{-1}d^{-1} = \mathcal{P} \circ 1 = h \circ \mathcal{P},$$

where $h \in H$, the group of covering transformations. Since $H$ is abelian,

$$\xi_{W_{ij}}^{\tilde{c}d^{-1} \tilde{d}^{-1}} = \xi_{W_{ij}}^{\tilde{d}^{-1} \tilde{c}^{-1}} = \xi_{W_{ij}}^{\tilde{d}^{-1} \tilde{c}^{-1}}$$

for all $i, j$, and so $\xi_{W_{ij}}^{\tilde{c}d} = (\xi_{W_{ij}})^{\tilde{c}d}$ for all $i, j$. Therefore, the whole group $G \cong M_0 \times N_0$ lifts to a group $\tilde{G}$, as desired. □
PROOF OF THEOREM 1.3. Let $\Gamma = K_{m,p^f}$, and let $\Delta_1$ be the vertex set of size $m$ and $\Delta_2$ be the vertex set of size $p^f$. First, by Proposition 5.2, $K_{2,p^f}$ has a locally 3-arc transitive cover. Suppose that the groups that lift are $M \times Z_2 \cong \mathrm{soc}(M)M_0 \times Z_2$, where $M$ acts 2-transitively on a set of size $m$ with point stabilizer $M_0$, and $N \times Z_2 \cong \mathrm{soc}(N):N_0 \times Z_2$, where $\mathrm{soc} N \cong Z_p^f$ and $N_0$ acts transitively on nonidentity elements of $\mathrm{soc}(N)$. We assign voltages from $H \cong Z_p^d$ based on these covers as in Proposition 5.6. In this case, the lift of $\mathrm{soc}(N)$ centralizes the voltages, and the way we have assigned voltages and extended the action of the lift of $\mathrm{soc}(N)$ to all of $H$ shows that the socle centralizes all of $H$. Moreover, the direct product of the vertex stabilizers $M_0 \times N_0$ also lifts by Proposition 5.6. To complete the proof, we must show that $M' \cong \mathrm{soc}(M)$ lifts, where $M'$ stabilizes every vertex in $\Delta_2$ and acts on $\Delta_1$ as $\mathrm{soc}(M)$ acts on $m$ points. If we let $\Delta_1 = \{\gamma_0, \ldots, \gamma_{m-1}\}$, $\Delta_2 = \{\delta_0, \ldots, \delta_{p^f-1}\}$, $W_{ij} = (\delta_0, \gamma_i, \delta_j, \gamma_0, \delta_0)$, and follow the notation as in the proof of Proposition 5.6 for the $c_i$ and $d_j$, for any $g \in M' \setminus M_0$, we need the following equations to be satisfied for all $i, j$:

$$(x^{c_i}d_j)(\tilde{g}) = \xi_{W_{ij}}^g = \xi_{W_{ij}} \mapsto (\delta_0, \gamma_i, \delta_j, \gamma_0, \delta_0)^g = (\delta_0, \gamma_i^g, \delta_j, \gamma_0^g, \delta_0),$$

and, if we let $\gamma_i^g = \gamma_k$ and $\gamma_0^g = \gamma_l$, we have:

$$(x^{c_i}d_j)(\tilde{g}) = x^{\tilde{c}_k}(x^{\tilde{c}_i})^{-1}.$$  

Since by hypothesis $g$ acts trivially on $\Delta_2$, $\tilde{g}$ commutes with $d_j$, and this last equation is equivalent to:

$$(x^{\tilde{c}_i}) = x^{\tilde{c}_k}(x^{\tilde{c}_i})^{-1}.$$  

Our choice of voltage assignment guarantees that this equation is satisfied, and proceeding as in the rest of the proof of Proposition 5.6 shows that indeed the group isomorphic to $M \times N$ lifts. Therefore, this cover is locally 3-arc transitive, as desired. \hfill \square

COROLLARY 5.7. For any prime $p$ and any $f \leq d \in \mathbb{N}$, the graph $K_{p^d,p^f}$ has a locally 3-arc transitive regular cover with covering transformation group $H \cong Z_p^d$.

The situation that is particularly tricky is dealing with $K_{m,n}$, where $K_{2,m}$ and $K_{2,n}$ both have locally 3-arc transitive covers with elementary abelian covering transformation group $H \cong Z_p^d$ and neither $m$ nor $n$ is 2 nor a power of $p$. Here, the socles of the groups that lift must act faithfully, and it is not always possible to get a locally 3-arc transitive cover in this situation, as the following lemma shows:

LEMMMA 5.8. Let $p \neq 3$ be a prime. Then $K_{3,3}$ does not have a locally 3-arc transitive regular cover with covering transformation group $Z_p \times Z_p$. 

Proof. It should be noted that this result is proved by different means in [1]. Label the vertices of $\Gamma = K_{3,3}$ as $\Delta_1 = \{1, 2, 3\}$ and $\Delta_2 = \{4, 5, 6\}$. Let $\xi_5 = x, \xi_5 = y, \xi_6 = x^i y^j$, and $\xi_6 = x^k y^l$, where $\langle x, y \rangle \cong Z_p \times Z_p$ and $0 \leq i, j, k, l \leq p-1$. Since $S_3$ is the only group that acts 2-transitively on three points, the automorphisms $g = (1 2 3)$ and $h = (4 5 6)$ must lift. The following equations must then be satisfied:

\[
x^{\bar{g}} \leftrightarrow (4, 1, 5, 2, 4)^g = (4, 2, 5, 3, 4) \leftrightarrow x^{-1} y
\]
\[
y^{\bar{g}} \leftrightarrow (4, 1, 5, 3, 4)^g = (4, 2, 5, 1, 4) \leftrightarrow x^{-1}
\]
\[
(x^i y^j)^{\bar{g}} \leftrightarrow (4, 1, 6, 2, 4)^g = (4, 2, 6, 3, 4) \leftrightarrow x^{k-i} y^{l-j}
\]
\[
(x^k y^l)^{\bar{g}} \leftrightarrow (4, 1, 6, 3, 4)^g = (4, 2, 6, 1, 4) \leftrightarrow x^{-i} y^{-j}
\]
\[
x^h \leftrightarrow (1, 5, 2, 4, 1)^h = (1, 6, 2, 5, 1) \leftrightarrow x^{i-1} y^j
\]
\[
y^h \leftrightarrow (1, 5, 3, 4, 1)^h = (1, 6, 3, 5, 1) \leftrightarrow x^k y^{l-1}
\]
\[
(x^i y^j)^{\bar{h}} \leftrightarrow (1, 6, 2, 4, 1)^h = (1, 4, 2, 5, 1) \leftrightarrow x^{-1}
\]
\[
(x^k y^l)^{\bar{h}} \leftrightarrow (1, 6, 3, 4, 1)^h = (1, 4, 3, 5, 1) \leftrightarrow y^{-1}.
\]

Noting that $(x^i y^j)^{\bar{g}} = (x^{\bar{g}})^i(y^{\bar{g}})^j = x^{-i-j} y^i$, we find that $x^{k+j} y^{l-j-i} = 1$. Proceeding similarly with $(x^k y^l)^{\bar{g}}, (x^i y^j)^{\bar{h}},$ and $(x^k y^l)^{\bar{h}}$, we have:

\[
x^{i-k-l} y^{k+j} = 1
\]
\[
x^{i^2-i-j} y^{j+i-j} = 1
\]
\[
x^{(i-1)k+kl} y^{j+k(l-1)+1} = 1.
\]

Hence we have the following equations that must be satisfied in $\text{GF}(p)$:

\[
k + j = 0
\]
\[
l - j - i = 0
\]
\[
i - k - l = 0
\]
\[
i^2 - i + jk + 1 = 0
\]
\[
j(i + l - 1) = 0
\]
\[
k(i + l - 1) = 0
\]
\[
l^2 - l + jk + 1 = 0
\]

Now, if $j = 0$, then $x^i y^j = x^i$, and by Theorem 1.1 $p = 3$, a contradiction. Hence $i = 1 - l$, and since $k = -j$ and $j = i - l$, the last equation becomes $l^2 - l - (2l - 1)^2 + 1 = 0$, i.e., $3l(l - 1) = 0$. If
l = 0, then \( j = -1, k = 1, \) and \( i = 1; \) if \( l = 1, \) then \( j = 1, k = -1, \) and \( i = 0. \) Switching the roles of \( x \) and \( y \) show that these are the same voltage assignment.

We will actually show something more general. Let \( H = \langle x, y \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_n, \) where \( n \geq 2. \) Assign the identity voltage to all edges of \( \Gamma \) incident with vertices 1 and 4, and we let \( \xi_{52} = x, \xi_{53} = \xi_{62} = y, \) and \( \xi_{63} = xy^{-1}. \) Call the resulting regular cover \( \tilde{\Gamma}. \) We claim that the automorphisms \( g = (1 \ 2 \ 3), h = (4 \ 5 \ 6), \) and \( f = (2 \ 3)(5 \ 6) \) lift, while the automorphisms \( (2 \ 3) \) and \( (5 \ 6) \) do not. It should be noted that this particular cover is actually a 2-arc transitive regular cover of \( K_{3,3} \) and was constructed by Conder and Ma in [1].

Clearly, \( \xi_{52}, \xi_{53}, \xi_{62}, \) and \( \xi_{63} \) form a generating set for the closed walks based at either 1 or 4. To see whether \( g \) lifts, we will use the closed walks based at 4. By Lemma 2.2, the following equations must be satisfied:

\[
\begin{align*}
x^g &\leftrightarrow (4, 1, 5, 2, 4) \leftrightarrow (4, 2, 5, 3, 4) \leftrightarrow x^{-1}y \\
y^g &\leftrightarrow (4, 1, 5, 3, 4) \leftrightarrow (4, 2, 5, 1, 4) \leftrightarrow x^{-1} \\
y^g &\leftrightarrow (4, 1, 6, 2, 4) \leftrightarrow (4, 2, 6, 3, 4) \leftrightarrow y^{-1}x^{-1}y = x^{-1} \\
(x^{-1}y)^g &\leftrightarrow (4, 1, 6, 3, 4) \leftrightarrow (4, 2, 6, 1, 4) \leftrightarrow y^{-1}.
\end{align*}
\]

Indeed, every group \( \mathbb{Z}_n \times \mathbb{Z}_n \) has such an automorphism. Similarly, using closed walks based at 1, the following equations must be satisfied for \( h \) to lift:

\[
\begin{align*}
x^h &\leftrightarrow (1, 5, 2, 4, 1) \leftrightarrow (1, 6, 2, 5, 1) \leftrightarrow yx^{-1} \\
y^h &\leftrightarrow (1, 5, 3, 4, 1) \leftrightarrow (1, 6, 3, 5, 1) \leftrightarrow x^{-1}yy^{-1} = x^{-1} \\
y^h &\leftrightarrow (1, 6, 2, 4, 1) \leftrightarrow (1, 4, 2, 5, 1) \leftrightarrow x^{-1} \\
(x^{-1}y)^h &\leftrightarrow (1, 6, 3, 4, 1) \leftrightarrow (1, 4, 3, 5, 1) \leftrightarrow y^{-1}.
\end{align*}
\]

Once again, every group \( \mathbb{Z}_n \times \mathbb{Z}_n \) has such an automorphism. Looking again at closed walks based at 1, the following equations must be satisfied in order for \( f \) to lift:

\[
\begin{align*}
x^f &\leftrightarrow (1, 5, 2, 4, 1) \leftrightarrow (1, 6, 2, 5, 1) \leftrightarrow yx^{-1} \\
y^f &\leftrightarrow (1, 5, 3, 4, 1) \leftrightarrow (1, 6, 3, 5, 1) \leftrightarrow x^{-1}yy^{-1} = x^{-1} \\
y^f &\leftrightarrow (1, 6, 2, 4, 1) \leftrightarrow (1, 4, 2, 5, 1) \leftrightarrow x^{-1} \\
(x^{-1}y)^f &\leftrightarrow (1, 6, 3, 4, 1) \leftrightarrow (1, 4, 3, 5, 1) \leftrightarrow y^{-1}.
\end{align*}
\]
\( x^f \leftrightarrow (1, 5, 2, 4, 1)^f = (1, 6, 3, 4, 1) \leftrightarrow x^{-1}y \)

\( y^f \leftrightarrow (1, 5, 3, 4, 1)^f = (1, 6, 2, 4, 1) \leftrightarrow y \)

\( y^f \leftrightarrow (1, 6, 2, 4, 1)^f = (1, 5, 3, 4, 1) \leftrightarrow y \)

\((x^{-1}y)^f \leftrightarrow (1, 6, 3, 4, 1)^f = (1, 5, 2, 4, 1) \leftrightarrow x \)

Once again, such an automorphism of \( Z_n \times Z_n \) exists, so all of \( f, g, h \) lift. Thus a group of automorphisms of \( K_{3,3} \) isomorphic to \((Z_3 \times Z_3):Z_2\) lifts, and the resulting cover will be locally 2-arc transitive since the vertex stabilizers act 2-transitively on neighbor sets.

On the other hand, look at \( t = (2 \ 3) \). In order for \( t \) to lift, we would need:

\( x^t \leftrightarrow (1, 5, 2, 4, 1)^t = (1, 5, 3, 4, 1) \leftrightarrow y \)

\( y^t \leftrightarrow (1, 5, 3, 4, 1)^t = (1, 5, 2, 4, 1) \leftrightarrow x \)

\( y^t \leftrightarrow (1, 6, 2, 4, 1)^t = (1, 6, 3, 4, 1) \leftrightarrow x^{-1}y \)

\((x^{-1}y)^t \leftrightarrow (1, 6, 3, 4, 1)^t = (1, 6, 2, 4, 1) \leftrightarrow y \)

However, since the group is not cyclic, \( x \neq x^{-1}y \), and so the equations have no solution. Hence the automorphism \( t \) does not lift, and so the fiber-preserving automorphisms of \( \tilde{\Gamma} \) cannot act locally 3-arc transitively.

**Proof of Theorem 1.4.** First, the result follows for \( K_{2,2}, K_{2,3}, K_{2,p}, K_{2,p^2}, K_{3,p}, K_{3,p^2}, K_{p,p}, K_{p,p^2}, \) and \( K_{p^2,p^2} \) by Theorems 1.1, 5.4, and 1.3. The only other possibility is \( K_{3,3} \), which is ruled out by Lemma 5.8.

**Proof of Corollary 1.5.** By Theorem 1.4 we only must consider \( K_{2,2}, K_{2,3}, K_{2,p}, K_{2,p^2}, K_{3,p}, K_{3,p^2}, K_{p,p}, K_{p,p^2}, \) and \( K_{p^2,p^2} \). However, \( K_{2,2} \) is just \( C_4 \), which has only one co-tree edge in a spanning tree, and can be eliminated immediately. The result follows from Corollaries 2.9 and 5.5 and reasoning as in the proof of Theorem 1.1.

It should be noted that none of the covers obtained in Theorems 1.1 and 1.4 were obtained from an almost simple group lifting. The following shows that, in fact, the complete bipartite graph \( \Gamma \) has a regular cover \( \tilde{\Gamma} \) with voltages from an elementary abelian group such that the full automorphism group of \( \Gamma \) lifts.

**Theorem 5.9 ([24]).** Let \( \Gamma \) be any connected graph with edge set \( E \), and let \( T \) be the edge set of any spanning tree of \( \Gamma \). Let \( H := Z_k^{[E \setminus T]} \),
and let \( \{ h_1, ..., h_{|E\setminus T|} \} \) be a minimal generating set for \( H \). Define \( \Gamma_k \) to be the derived covering graph of \( \Gamma \) obtained by assigning a distinct \( h_i \) to each edge in \( E \setminus T \). Then the full automorphism group \( \text{Aut}(\Gamma) \) lifts to \( \text{Aut}({\tilde \Gamma}) \).

**Proof.** Although this theorem is not expressly stated in this manner in [24] (only specific types of automorphisms and a specific class of graphs called polygonal graphs were considered), the proof of [24, Theorem 1.3] carries through to an arbitrary automorphism of an arbitrary graph.

On the other hand, a group \( G \) having a degree \( d \) representation over \( GF(p) \) does not actually guarantee that a voltage assignment from \( Z_p^d \) exists such that \( G \) lifts.

**Example 5.10.** Even though \( A_5 \lesssim \text{GL}(3, 11) \), there is no voltage assignment from \( Z_p^3 \) to \( K_{2,5} \) such that \( A_5 \) lifts.

To see this, we let \( \Gamma = K_{2,5} \) with vertex sets \( \Delta_1 = \{ \alpha, \beta \} \) and \( \Delta_2 = \{ 0, 1, 2, 3, 4 \} \). Without a loss of generality, we may set \( \xi_{1\beta} = w, \xi_{2\beta} = x, \xi_{3\beta} = y, \) and \( \xi_{4\beta} = w^i x^j y^k \), where \( x, y, z \leq Z_p^3 \) for some prime \( p \) and \( 0 \leq i, j, k \leq p - 1 \). Note that since \( A_5 \) has no degree 2 representation over \( GF(p) \), \( p \) prime, we may assume that \( \langle w, x, y \rangle \cong Z_p^3 \). Now, \( A_5 \) will lift only if the automorphisms \( g = (0\ 1\ 2\ 3\ 4) \) and \( h = (2\ 3\ 4) \) that generate \( A_5 \) lift, which we assume fix \( \Delta_1 \); we know that the automorphism fixing \( \Delta_2 \) and swapping \( \alpha \) and \( \beta \) lifts. By Lemma 2.2, there must exist \( \tilde{g} \) and \( \tilde{h} \) satisfying the following equations:

\[
\begin{align*}
w^{\tilde{g}} &= xw^{-1} \\
x^{\tilde{g}} &= yw^{-1} \\
y^{\tilde{g}} &= w^i x^j y^k \\
(w^i x^j y^k)^{\tilde{g}} &= w^{-1} \\
w^{\tilde{h}} &= w \\
x^{\tilde{h}} &= y \\
y^{\tilde{h}} &= w^i x^j y^k \\
(w^i x^j y^k)^{\tilde{h}} &= x
\end{align*}
\]

Now, \( w^{-1} = (w^i x^j y^k)^{\tilde{g}} = (w^{\tilde{g}})^i (x^{\tilde{g}})^j (y^{\tilde{g}})^k \), which implies that \( w^{1-i-j+k(i-1)} x^{i+j+k} y^{j+k^2} = 1 \). Since \( \langle w, x, y \rangle = Z_p^3 \), the only way this can happen is if \( 1 - i - j + k(i - 1) = i + j k = j + k^2 = 0 \).
in GF(p). Hence \( j = -k^2 \), and then \( i = k^3 \), which in turn implies that \( k^4 - k^3 + k^2 - k + 1 = 0 \) in GF(p). On the other hand,

\[
x = (w^3 x^j y^k) = (w^3 x)(x^k y)(y^k) = (w x^k y)(x^k y) - k^2 (y^k),
\]

which implies that \( x k^3 + 1 = w^{k^3(k+1)} \). Again, the assumption that \( \langle w, x, y \rangle = Z_p \) implies that \( k^3 + 1 = k^3(k+1) = 0 \) in GF(p), which implies that \( k = p - 1 \). Plugging this back into the equation \( k^4 - k^3 + k^2 - k + 1 = 0 \) implies that \( 5 = 0 \) in GF(p), i.e., \( p = 5 \). Hence there is no voltage assignment from \( Z_3 \) to the edges of \( K_{2,5} \) such that \( A_5 \) lifts.

It should be noted that (using [10] and [19]), if \( w, x, y \) are generators for \( Z_3 \) and \( \Gamma = K_{2,5} \) with vertex sets \( \Delta_1 = \{\alpha, \beta\} \) and \( \Delta_2 = \{0, 1, 2, 3, 4\} \), then assigning \( \xi_{1,\beta} = w, \xi_{2,\beta} = x, \xi_{3,\beta} = y, \) and \( \xi_{4,\beta} = w^{-1} x^{-1} y^{-1} \) yields a regular cover where the whole automorphism group of \( K_{2,5} \) lifts.

6. Application to locally 2-arc transitive regular covers of complete bipartite graphs

This paper is merely the tip of the iceberg when it comes to locally s-arc transitive covers of complete bipartite graphs, and it would be very interesting to extend these results to locally 2-arc transitive regular covers. This is a much harder problem. As demonstrated in the proof of Lemma 5.8, while \( K_{3,3} \) has no locally 3-arc transitive regular covers with covering transformation group isomorphic to \( Z_p \times Z_p \), \( p \neq 3 \) a prime, it does have a locally 2-arc transitive regular cover with this covering transformation group. Moreover, the lemmas developed in Section 2 simply do not apply.

However, there is hope for the locally 2-arc transitive case. In a recent paper [8], the minimal groups \( G \) such that a complete bipartite graph is locally \((G,2)\)-arc transitive were completely classified. Any locally 2-arc transitive regular cover of a complete bipartite graph that is not a locally 3-arc transitive regular cover must come from one of the graphs listed in [8]. Hence the results of this paper, combined with a careful analysis of the graphs and groups listed in [8], provide hope for classification results for locally 2-arc transitive covers of complete bipartite graphs.

References

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